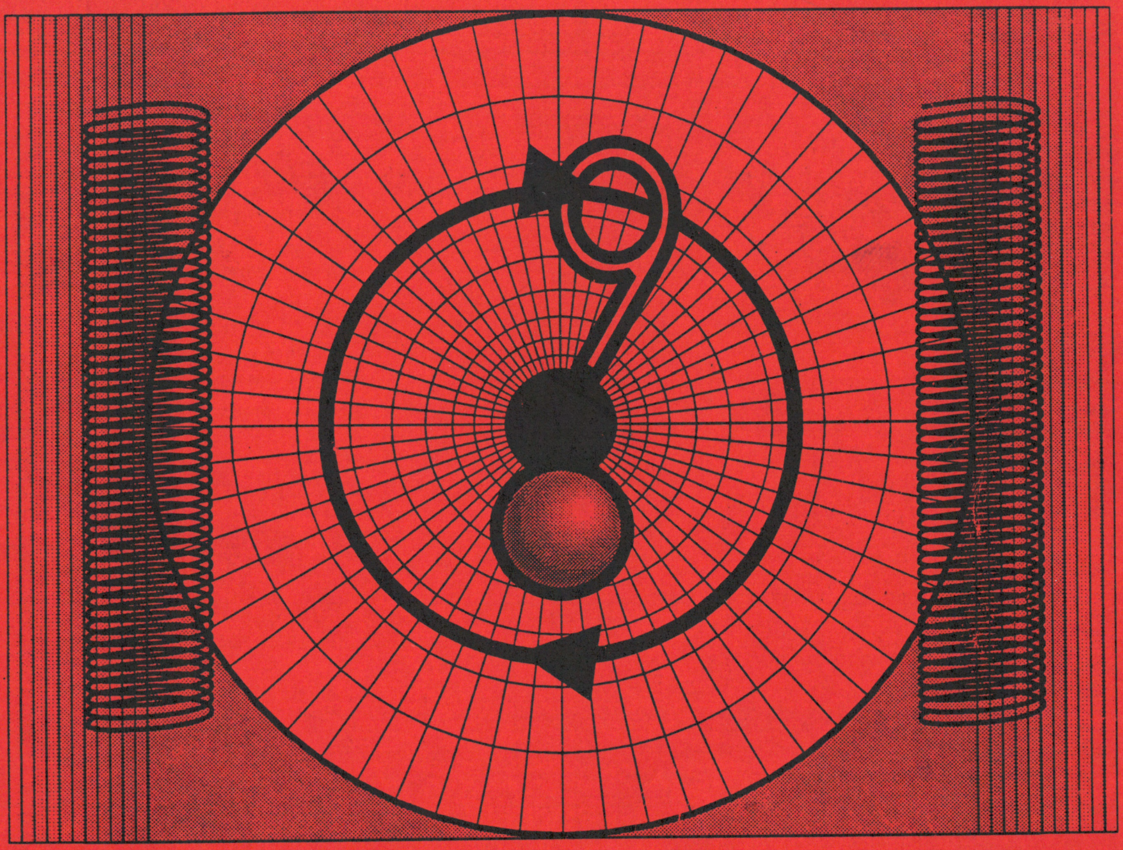
ELECTROMAGNETIC FIELD THEORY

A Student's Manual

G.M.Tattersfield



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G. M. Tattersfield

(1998 Edition)

Introduction

This handbook has been written to accompany the third-year module in Electromagnetic Engineering, given at the University of Cape Town. The main motivation for providing these notes in handbook form has been that students come to the course with a very wide variety of educational backgrounds; in particular, there are considerable differences in the amount of formal physics that they have covered. The notes for the course therefore aim to provide a description of electromagnetic field theory which can be used either as a first introduction or as a coherent revision programme for the subject. The emphasis is necessarily on the use of electromagnetic field theory by electrical engineers, and so the topics chosen for coverage have always been with that slant in mind.

Throughout the notes, it is assumed that students have a working familiarity with the vector algebra that is studied as a core course in the second year at UCT. It is *not* assumed that students have taken second year physics, however. Only the first-year physics course is a prerequisite.

The notes are presented in an open format, to encourage the writing of further comments in the margins. This is in the belief that it is better for students to be able to concentrate on the material during lectures, rather than feeling that they need to write copiously. The basic outlines of the lectures are laid down in the notes, and students are invited to read up in advance of each lecture, and to add to the handbook only where they feel it necessary. Much of this should preferably be done after the lecture, with reference to some of the many textbooks that are available on the subject.

At the back of the handbook there are a few test and examination questions, which were given as part of the course in 1995 and 1996. They are included as a sample, indicating the standard required at the end of the course.

I am very grateful to Richard Lord for helping to compile the text in LaTeX, and for producing the diagrams. The material for these notes is derived from several textbooks, but errors in interpretation are, of course, my own. I am always very grateful to be informed of any errors encountered in the use of the handbook so that they can be corrected in later editions.

To the Student

This set of notes is designed to help you in your study of electromagnetic fields as part of the third year course at UCT. It is very important, however, that you appreciate the spirit in which the notes were produced: they are *not* intended to substitute for lectures, and they should stimulate you to look for more information in textbooks. Textbooks will give you much more information than is possible in a short handbook of this nature, and will enrich your understanding of the subject, particularly if you make a point of dipping into several works of references from time to time.

It is also important for you to study this material on a regular and continuous basis, rather than to try to learn it in short bursts before tests! These notes are designed to help you to work steadily on improving your understanding, going forward at the rate of about one chapter per lecture. You may find that you need to brush up on your second-year maths. Put some time into this early in the course, if necessary—it will enhance your enjoyment of the later chapters!

Field theory is a highly theoretical part of the background education of any electrical engineer. The fundamentals of field theory help us to understand many phenomena at a basic, physical level. However, there can be few practical experiments in a course of this nature, and students often find it difficult to come to terms with the large amount of theory. Persevere, and the course will offer you explanations of many matters that are relevant to electrical engineers. Among these, we will tackle the following questions:

- What exactly is meant by voltage and current?
- What is the theoretical basis of capacitance and inductance, and how can they be calculated?
- How do electric and magnetic fields store and release energy?
- Do Maxwell's equations describe all that we know about electromagnetic phenomena?

- Why do electromagnetic waves travel through a vacuum at a velocity given by the well-known constant c?
- Why are metals shiny, and why are radio waves reflected from layers in the ionosphere?

As you can see from this short list, we will address some very fundamental issues in electrical engineering. I hope that you will enjoy using this handbook as part of your studies in this area, and that the skills which you gain in taking this course will be of value to you later in your career.

G.M.T.

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Part 1

Electrostatics

Chapter 1

Electric Fields

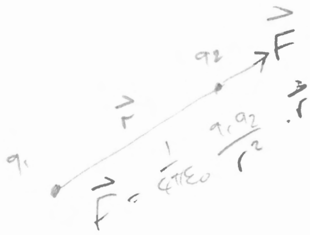
1.1 What is a field?

- A way to visualise a measured quantity mathematically, in a region with several dimensions.
- e.g. Temperature in a room - a 3D *scalar* field;
  e.g. Pressure on the blackboard - a 2D *scalar* field;
  e.g. Velocity of a river - a 3D *vector* field.
- Typically, some other physical quantity is the *source* of the field. Thus:
  - Sources of heat give rise to the temperature in the room;
  - Writing on the blackboard is a source of pressure;
  - Changes in the height of a river's bed determine its velocity;
  - *Mass* is the source of gravitational fields in space.
- The source of electric field is *charge*.

1.2 The force between charges

- Everyone has seen the force between charges in action:
  - A comb attracts small pieces of paper in dry conditions;

  - A balloon sticking to a wall;
  - Xerography.
- The fundamental experimental observation of electromagnetic theory:
  - Like charges repel and unlike charges attract;
  - The force between two charges in a vacuum was found by Coulomb to be



$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \cdot \vec{r}$

$\vec{r} = \frac{\vec{r}}{|\vec{r}|}$

$$f = K\frac{q_1 q_2}{\epsilon_o r^2} \text{ (in magnitude) or } \mathbf{f} = K\frac{q_1 q_2}{\epsilon_o r^2}\hat{\mathbf{r}} \text{ (in vector form)} \tag{1.1}$$

where

  * q_1,q_2 are the magnitudes of the two charges in coulombs;
  * r is their distance apart in metres;
  * $\hat{\mathbf{r}}$ is the unit vector from one charge to the other;
  * ϵ_o (from experiment) is $8.854 \times 10^{-12} \approx \frac{1}{36\pi} \times 10^{-9}$ F/m;
  * K is chosen to be $\frac{1}{4\pi}$ to ensure that f is in newtons;
  * In the more general case where the charges are embedded in a medium, ϵ is found from $\epsilon_o\epsilon_r$, where ϵ_r depends on the material separating the charges and is called the *relative permittivity* or *dielectric constant* of the material.

1.3 Electric field

- In the vicinity of a charge (or group of charges) we are often interested to know how other charged objects will behave. e.g. The behaviour of an electron in a TV tube with highly-charged X and Y plates.
- To do this, we visualise the *force per unit charge* for each point in the region of interest: this is called the *electric field*. Thus:

$$\mathbf{E} \triangleq \frac{\mathbf{f}}{\Delta \mathrm{q}} \tag{1.2}$$

for a tiny *test charge*, Δq, too small to contribute significantly to the field in its own right.

• Δq (test charge)

- Hence, the electric field of a single point of charge, q, is given by

$$\mathbf{E} = K\frac{q\Delta q}{\epsilon r^2 \Delta q}\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon r^2}\hat{\mathbf{r}} \tag{1.3}$$

NC^{-1}

Vm^{-1}

• q

  Note: Since $\hat{\mathbf{r}}$ is a unit radius vector from q, the E-field points away from q in all directions, if q is a positive charge, or towards it if q is negative.

- Recalling that charge, capacitance and voltage are related by $Q = CV$, we see from the previous two equations that the units of the electric field are either N/C or (C)(m/F)(1/m^2) =V/m.

1.4 Superposition of fields

- A *principle of superposition* holds for finding the electric field due to two or more charges - simply add the fields from each individual charge vectorially.

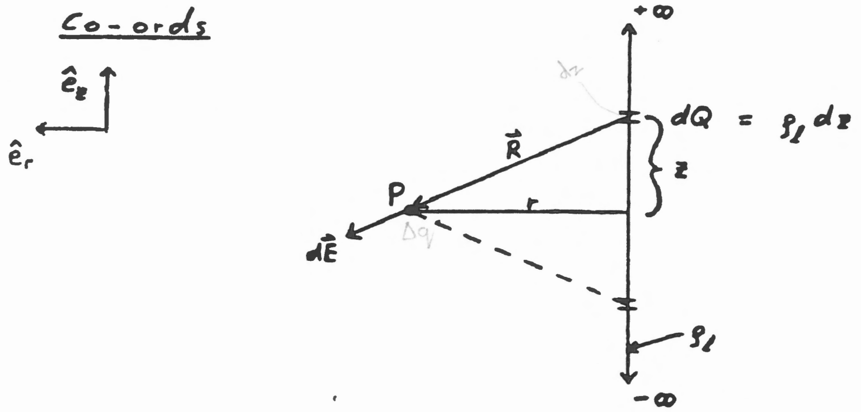
- Thus, the field due to two point charges q_1 and q_2 is simply

$$\mathbf{E} = \frac{q_1}{4\pi\epsilon r_1^2}\hat{\mathbf{r}}_1 + \frac{q_2}{4\pi\epsilon r_2^2}\hat{\mathbf{r}}_2 \tag{1.4}$$

  where "+" denotes *vector* addition.

- We often want to know the field due to whole lines or sheets or clouds of charge. To do this, we need to know the *charge density*, ρ_l (in C/m) or ρ_S (in C/m^2) or just ρ (in C/m^3) and the *element* of the group of charges, dl or dS or dV. We must then *integrate* the field due to each elemental charge, $\rho_l dl$, $\rho_S dS$ or ρdV as appropriate.

e.g. The field of an infinite line charge



- Let the charge density all along the line be ρ_l in Cm^{-1}
- The elemental charge is therefore $dQ = \rho_l dz$
- For a test charge at the general point P, Coulomb's Law gives:

$$d\mathbf{E} = \frac{d\mathrm{f}}{\Delta Q} = \frac{dQ\Delta Q}{4\pi\epsilon_o R^2 \Delta Q}\hat{\mathbf{R}} = \frac{dQ}{4\pi\epsilon_o R^2}\hat{\mathbf{R}} = \frac{dQ}{4\pi\epsilon_o R^2}\frac{r\hat{\mathbf{e}}_r - z\hat{\mathbf{e}}_z}{\sqrt{r^2+z^2}} \tag{1.5}$$

- Integrating along the full length of the line of charge, all z-components cancel, so (using integration tables) the total E-field is calculated as:

by symmetry

$$\mathbf{E} = \int_{-\infty}^{\infty} \frac{\rho_l r dz}{4\pi\epsilon_o (r^2+z^2)^{3/2}}\hat{\mathbf{e}}_r = \frac{\rho_l r}{4\pi\epsilon_o}\left[\frac{z}{r^2\sqrt{r^2+z^2}}\right]_{-\infty}^{\infty}\hat{\mathbf{e}}_r = \frac{\rho_l}{2\pi\epsilon_o r}\hat{\mathbf{e}}_r \tag{1.6}$$

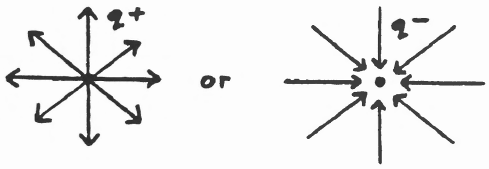
1.5 Visualising an electric field

- When you get answers like the above, it is important to *interpret* them. We can do this with a sketch of the E-field.

  e.g.1 E-field of a single point :

  $$\mathbf{E} = \frac{q}{4\pi \epsilon r^2}\hat{\mathbf{r}}$$

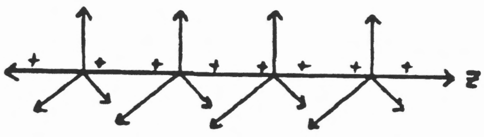
  Remember that the E-field is defined as the force per unit charge acting on a *positive* test charge.



  e.g.2 E-field of an infinite line charge :

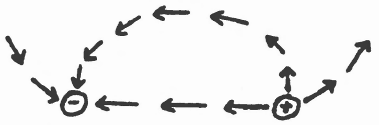
  $$\mathbf{E} = \frac{\rho_l}{2\pi \epsilon_o r}\hat{\mathbf{e}}_r$$

  Note that the field is *not* inverse-square.



  e.g.3 E-field of a pair or group of charges :

  Here you have to draw *electric field lines* that are everywhere tangential to the total electric field vector.

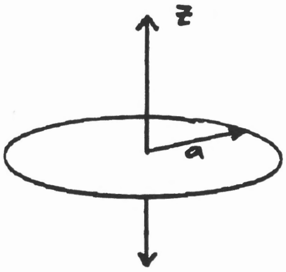
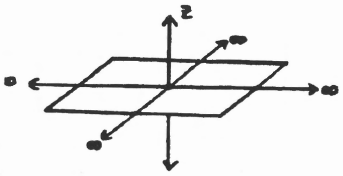
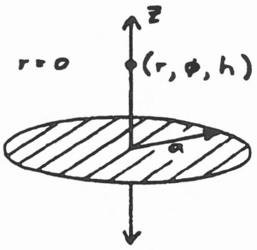


- Try experimenting with the iron-filings program to see how to arrive at the sketch of the field in example 3 above. Remember, however, that we are dealing with electrostatic fields, not magnetic fields, here - so the concept of iron filings is only an analogy.

1.6 Fields of standard charge configurations

Apart from the infinite line charge, there are other standard charge configurations whose fields you should be able to calculate and draw. You should do your own research (work in small groups) to make sure that you can clearly demonstrate the calculations leading to the results on the following page.

- Can you see how 3 → 2 as $a \to \infty$?

Configuration	E-field	Sketch of the configuration
1. A ring of charge of radius a, in the plane perpendicular to the z-axis	$\frac{\rho_l a z}{2\epsilon(a^2+z^2)^{3/2}}\hat{\mathbf{e}}_z$ (on the z-axis)	
2. An infinite plane of charge, filling the x-y plane	$\frac{\rho_S}{2\epsilon}\hat{\mathbf{e}}_z$	
3. A uniformly-charged circular disc of radius a, in the plane perpendicular to the z-axis	$\frac{\rho_S h}{2\epsilon}\left(\frac{-1}{\sqrt{a^2+h^2}}+\frac{1}{h}\right)\hat{\mathbf{e}}_z$ (at $(0,\phi,h)$ on the z-axis)	

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Chapter 2

Electric Flux Density

2.1 Electric Flux and Flux Density

- So far we have dealt with **E** in V/m, which is strictly called the *Electric Field Intensity.* We have drawn *field lines* to illustrate electric fields. We need two more very important concepts for a much better way to handle fields.

- *Electric Flux* (Ψ) originates on positive charges and we visualise it as flowing along electric field lines towards negative charges (a sort of electric stream).
  Note: Flux is just another way of thinking about charge (1C of charge $\equiv$ 1C of flux or $\Psi = Q$). If this seems strange, think of light, which we often conceptualise in the same way.

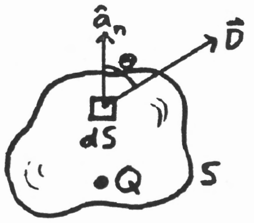
- Electric flux is a *scalar*, but we are often interested in how many flux lines flow through a given area, and which way they are flowing.

  If near P in the small area dS, normal to $\hat{\mathbf{a}}_n$, flux $d\Psi$ passes through dS parallel to $\hat{\mathbf{a}}_n$, we then think of the *electric flux density* at P as being

$$\mathbf{D} = \frac{d\Psi}{dS}\hat{\mathbf{a}}_n \qquad (\text{in C/m}^2) \tag{2.1}$$

2.2 Gauss's Law

- Now consider an *arbitrary* charge Q. It may be a point charge or some linear, surface or volume distribution—it doesn't matter. We *surround* it with some arbitrary surface, S, whose actual shape doesn't matter either (we call this a gaussian surface).
- At an arbitrary point P on S, the flux density $\mathbf{D}$ through dS near P will almost certainly not be parallel to $\hat{\mathbf{a}}_n$, the normal to dS. Instead $\hat{\mathbf{a}}_n$ and $\mathbf{D}$ will have some angle, θ, between them.
- By our definition, the flux through dS will be



$$
\begin{aligned}
d\Psi &= |\mathbf{D}|\, dS \cos\theta && (2.2)\\
&= \mathbf{D}\cdot dS\,\hat{\mathbf{a}}_n \qquad \text{since } \mathbf{D}\cdot\hat{\mathbf{a}}_n = |\mathbf{D}||\hat{\mathbf{a}}_n|\cos\theta = |\mathbf{D}||1|\cos\theta && (2.3)\\
&= \mathbf{D}\cdot d\mathbf{S} && (2.4)
\end{aligned}
$$

where $d\mathbf{S}$ is the *vector surface element* which has magnitude dS and is in the direction of $\hat{\mathbf{a}}_n$.

- Integrating over the whole closed surface S now gives us

$$\Psi = \oint_S \mathbf{D}\cdot d\mathbf{S} \tag{2.5}$$

and, since $\Psi = Q$, we get

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q \tag{2.6}$$

or, for a charge distribution

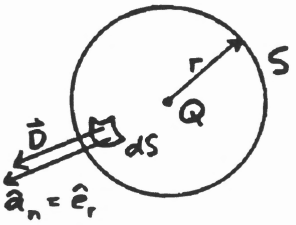
$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dV \tag{2.7}$$

This remarkable result, known as *Gauss's Law*, says that *the total flux out of a closed surface is equal to the charge enclosed by the surface.*

2.3 Advantage of Gauss's Law

- Gauss's Law enables us to calculate electric fields using the *electric flux density*, **D**, and not *electric field intensity*, **E**, directly. We look quickly at why this is an advantage.

- Consider a charge Q at the centre of a sphere S (a gaussian surface of radius r). From Gauss's Law,



$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} \tag{2.8}$$

$$= D \oint_S dS \tag{2.9}$$

$$= D(4\pi r^2) \tag{2.10}$$

Hence $D = \dfrac{Q}{4\pi r^2}$ so $\mathbf{D} = \dfrac{Q}{4\pi r^2}\hat{\mathbf{e}}_r$ where $\hat{\mathbf{e}}_r$ is the normal to dS on the sphere (and may also be written as the unit radial vector $\hat{\mathbf{r}}$).
We also know that $\mathbf{E} = \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{e}}_r$ from the previous lecture, so evidently in this case, as in every other,

$$\mathbf{D} = \epsilon\mathbf{E} \tag{2.11}$$

where ϵ is the *permittivity* of the medium.

- In the vicinity of a charge, therefore, **D** is *independent of the medium* (or dielectric) but **E** is not. In problems involving multiple dielectrics it is a great advantage to calculate **D** from the given charge and then convert to **E** in the end dielectric.

- Remember: *Electric field*, **E**, and Electric Flux Density, **D**, are both vector fields that have the same form and differ only by the constant factor ϵ, determined by the medium.

2.4 Polarisation of the Medium

- Before starting to use Gauss's Law, we ought to understand *why* the electric field around a charge is affected by the dielectric medium, and *how* we quantify the effect.

- Many media consist of atoms or molecules that are *polar*: they have distinct positive and negative regions (like the nucleus and electrons of an atom). A charge placed in such a medium tends to attract the regions of unlike charge, twisting or distorting the atoms or molecules of the medium. This is called *polarisation*.
  Note that polarisation tends to counteract the field due to the point charge.

- We account for polarisation by writing $\mathbf{D} = \epsilon_o\mathbf{E}+\mathbf{P}$, where $\mathbf{P}$ measures the extra effect on $\mathbf{D}$ of this *electric polarisation*. Furthermore, in most materials the polarisation, $\mathbf{P}$, is *linearly* related to the applied $\mathbf{E}$-field. We choose to write this as:

  $$\mathbf{P} = \epsilon_o\chi_e\mathbf{E} \tag{2.12}$$

  We then have the two equations:

  $$\begin{aligned}\mathbf{D} &= \epsilon_o(1+\chi_e)\mathbf{E} && (2.13)\\ &= \epsilon\,\mathbf{E} && (2.14)\end{aligned}$$

- Hence, since $\epsilon = \epsilon_r\epsilon_o$, we can write the *relative permittivity* of the medium as

  $$\epsilon_r = 1 + \chi_e \tag{2.15}$$

  The constant χ_e is called the *electric susceptibility* of the material of the medium.

e.g.1 The field of an infinite line charge

- Construct a *special gaussian surface* around the charge (i.e. a *closed* surface at which **D** from the charge is either *normal and constant* or *tangential*. Since the z-component of **D** is zero in this case, the surface is a closed circular right cylinder:
- By Gauss's Law:

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_1 \mathbf{D} \cdot d\mathbf{S}_1 + \int_2 \mathbf{D} \cdot d\mathbf{S}_2 + \int_3 \mathbf{D} \cdot d\mathbf{S}_3 \tag{2.16}$$

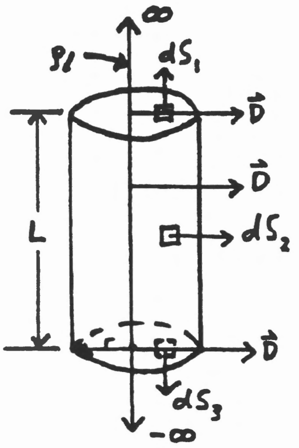
So

$$\rho_l L = 0 + D(2\pi r L) + 0 \tag{2.17}$$

Therefore

$$\begin{aligned} |\mathbf{D}| &= \frac{\rho_l L}{2\pi r L} && (2.18) \\ &= \frac{\rho_l}{2\pi r} && (2.19) \end{aligned}$$

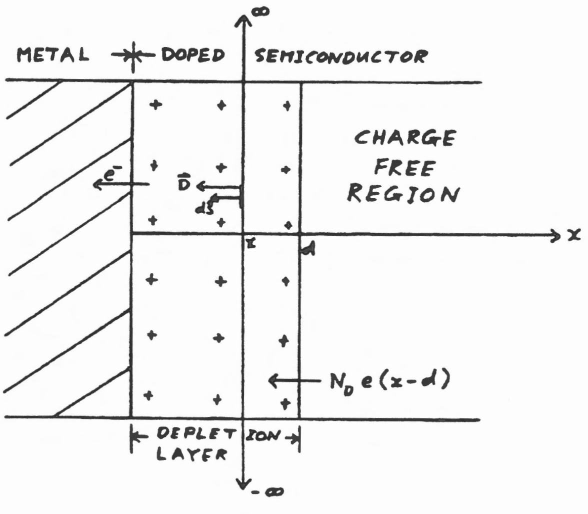
Thus $\mathbf{D} = \dfrac{\rho_l}{2\pi r}\hat{\mathbf{e}}_r$ and so $\mathbf{E} = \dfrac{\rho_l}{2\pi \epsilon r}\hat{\mathbf{e}}_r$ (as before but much more simply).



e.g.2 The field in a semiconductor depletion layer

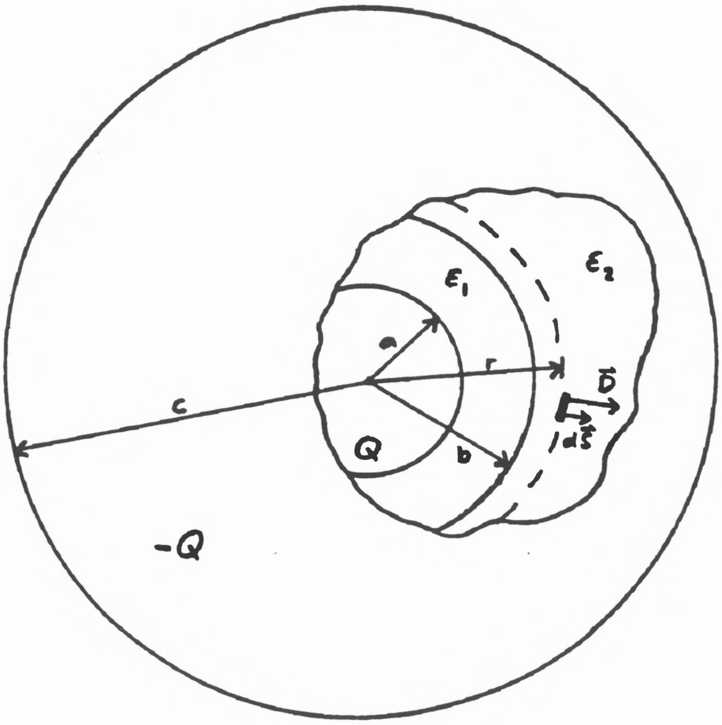
- An important and interesting practical example. Doped semiconductor in intimate contact with metal (or semiconductor of opposite doping as in the PN junction) gives rise to a depletion layer for $0 < x < d$ as shown.
- In the $x > d$ region, there is no net charge, and so $Q = 0$. Therefore $\mathbf{E} = 0$ and hence $\mathbf{D} = 0$ in this region, so all flux begins on the positive donor nuclei in the depletion layer and must end at the metal surface.
- The special gaussian surface for this problem includes the plane $x = x'$, where $0 < x' < d$. The surface is closed, but only returns to $x = x'$ at great distance from the depletion layer, so that effectively no flux passes through the curved return path of the surface.
- By Gauss's Law, $Q = \oint \mathbf{D} \cdot d\mathbf{S}$ so $N_D e(x - d) = D_x(x)$. where N_D represents the number of donor nuclei per unit volume in the depletion layer, and e is the charge on a single electron. Note that we have already divided through by the cross-sectional area of the semiconductor device, and have now written the flux density as a function of x.

Hence, the x-component of electric field is $E_x = \dfrac{N_D e(x - d)}{\epsilon}$.



e.g.3 The field between concentric spherical electrodes

- Note that the dielectric medium between the spheres has permittivity ϵ_1 for $a < r < b$, and permittivity ϵ_2 for $b < r < c$.
- The spherical symmetry implies that the electric field will be *radial.*
- Let the charge on the inner sphere be Q and the charge on the outer sphere be $-Q$.
- The special gaussian surface for the problem is a sphere, concentric with the other spheres, of radius $a < r < b$ for medium 1 or $b < r < c$ for medium 2.
- By Gauss's Law $Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = (D_r)(4\pi r^2)$. Hence $D_r = \dfrac{Q}{4\pi r^2}$ anywhere between the two charged spheres.
- The E-fields now follow easily: $E_r = \dfrac{Q}{4\pi\epsilon_1 r^2}$ for $a < r < b$ and $E_r = \dfrac{Q}{4\pi\epsilon_2 r^2}$ for $b < r < c$. So note that the **D**-field is continuous at $r = b$, but that the **E**-field changes abruptly there.



Chapter 3

Electrostatic Potential

3.1 The conservative property

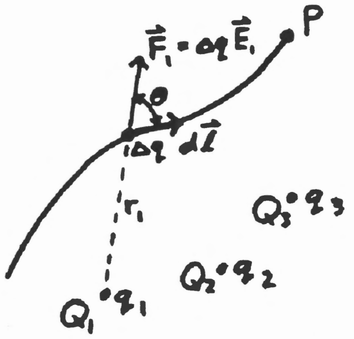
- A charge, Q, placed in an electrostatic field, $\mathbf{E}$, experiences a force $\mathbf{F} = Q\mathbf{E}$. If the charge were to move within the field, it would therefore either gain energy or lose it, depending on whether it moved with or against the local value of the force.

- In a field, $\mathbf{F}$, whose strength varies in space, the differential work done on Q by the field as Q moves along vector $d\mathbf{l}$ is written

$$dW = F dl \cos\theta = \mathbf{F} \cdot d\mathbf{l} \tag{3.1}$$

  Since total energy is conserved, this movement of Q also does work $dW = -\mathbf{F} \cdot d\mathbf{l}$ on the system of charges that give rise to the field $\mathbf{E}$.

- Consider the work done in moving a test charge Δq from infinity to P, a point in the vicinity of a cluster of charges $q_1, q_2, q_3 \ldots$ at points $Q_1, Q_2, Q_3 \ldots$. Arising from the movement of Δq, the total work done on the system (i.e. on $q_1, q_2, q_3 \ldots$) is given on the next page, in terms of the diagram below.



bit of geometry involved here

$$
\begin{aligned}
W &= \sum_n \left[-\int_\infty^{PQ_n} \mathbf{F}_n \cdot d\mathbf{l} \right] \\
&= -\int_\infty^{PQ_1} \mathbf{F}_1 \cdot d\mathbf{l} - \int_\infty^{PQ_2} \mathbf{F}_2 \cdot d\mathbf{l} - \int_\infty^{PQ_3} \mathbf{F}_3 \cdot d\mathbf{l} - \dots \\
&= -\int_\infty^{PQ_1} \frac{\Delta q q_1 \overbrace{\cos\theta_1 dl}^{dr_1}}{4\pi\epsilon r_1^2} - \int_\infty^{PQ_2} \frac{\Delta q q_2 \overset{dr_2}{\cos\theta_2 dl}}{4\pi\epsilon r_2^2} - \int_\infty^{PQ_3} \frac{\Delta q q_3 \overset{dr_3}{\cos\theta_3 dl}}{4\pi\epsilon r_3^2} - \dots \\
&= \left[\frac{\Delta q q_1}{4\pi\epsilon r_1} \right]_\infty^{PQ_1} + \left[\frac{\Delta q q_2}{4\pi\epsilon r_2} \right]_\infty^{PQ_2} + \left[\frac{\Delta q q_3}{4\pi\epsilon r_3} \right]_\infty^{PQ_3} + \dots \quad (\text{since } dr_1 = dl\cos\theta_1 \text{ etc.}) \\
&= \frac{\Delta q q_1}{4\pi\epsilon PQ_1} + \frac{\Delta q q_2}{4\pi\epsilon PQ_2} + \frac{\Delta q q_3}{4\pi\epsilon PQ_3} + \dots \qquad (3.2)
\end{aligned}
$$

- So the total work done on the system *depends only on the final position, P and not on the path taken by the charge,* Δq. Hence, if the charge moves around any closed path, no net work is done, so we can write

$$\oint \mathbf{F} \cdot d\mathbf{l} = 0 \quad \text{and so finally} \quad \oint \mathbf{E} \cdot d\mathbf{l} = 0 \qquad (3.3)$$

This is known as the *conservative property* of an electrostatic field.

3.2 The electrostatic potential

- When a charge Q moves from point B to point A inside an electrostatic field $\mathbf{E}$, the work done per unit charge is called the *potential of point A with respect to point B*. This is variously written

$$\Phi_A - \Phi_B = V_{AB} = W/Q = -\int_B^A \mathbf{E} \cdot d\mathbf{l} \qquad (3.4)$$

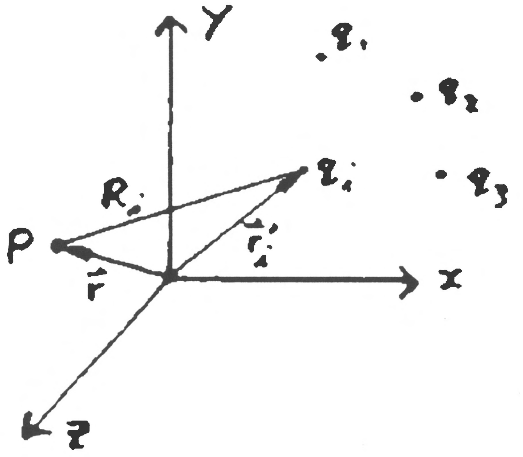
and the unit of the potential is thus the joule/coulomb or *volt*.

- Note:
  1. Strictly, we only define *potential difference*. The "potential at B", $\Phi_B = V_B$, is actually arbitrary, and a point of zero potential may be placed anywhere for convenience.
  2. For three points, A, B and C within $\mathbf{E}$, since $\mathbf{E}$ is conservative we have $V_{AB} = V_{AC} - V_{BC}$.
  3. By our defining equation, $V_{AB} = -V_{BA}$.
  4. Depending upon the co-ordinate system in use in a particular problem, the expressions for $d\mathbf{l}$ are:
     (a) $d\mathbf{l} = dx\hat{\mathbf{e}}_x + dy\hat{\mathbf{e}}_y + dz\hat{\mathbf{e}}_z$ in cartesian co-ordinates;
     (b) $d\mathbf{l} = dr\hat{\mathbf{e}}_r + rd\phi\hat{\mathbf{e}}_\phi + dz\hat{\mathbf{e}}_z$ in cylindrical co-ordinates;
     (c) $d\mathbf{l} = dr\hat{\mathbf{e}}_r + rd\theta\hat{\mathbf{e}}_\theta + r\sin\theta d\phi\hat{\mathbf{e}}_\phi$ in spherical co-ordinates.
- If the potential at infinity is taken as zero (often but not always convenient) then the potential of the test charge Δq at the general point P is simply

$$\frac{W}{\Delta q} = \frac{q_1}{4\pi\epsilon PQ_1} + \frac{q_2}{4\pi\epsilon PQ_2} + \frac{q_3}{4\pi\epsilon PQ_3} + \ldots = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon R_i} \qquad (3.5)$$

where $PQ_i = R_i = |\mathbf{r} - \mathbf{r}_i'| = \sqrt{(x - x_i')^2 + (y - y_i')^2 + (z - z_i')^2}$

Note how the prime superscript denotes a *source* of electrostatic field



- If the source of the field is a *continuous* distribution of charge, of density $\rho(\mathbf{r}')$ within a volume V, the *potential function* of the distribution is

  $\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')dV'}{4\pi\epsilon R}$ if the potential at infinity is zero, or

  $\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')dV'}{4\pi\epsilon R} + C$ if zero potential is defined as being elsewhere.

  (C is just a constant)

e.g.1 The potential of a point charge, Q

Since the field is known to be radially symmetric ($\mathbf{E} = \frac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}$), we work in *spherical* co-ordinates with $d\mathbf{l}$ replaced by using

$$d\mathbf{l} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\,\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \tag{3.6}$$

The potential is then

$$\begin{aligned} V_{AB} &= -\int_B^A \mathbf{E}\cdot d\mathbf{l} = -\int_{r_B}^{r_A} E_r\cdot\hat{\mathbf{e}}_r\,dr = -\int_{r_B}^{r_A} E_r\,dr & (3.7)\\ &= -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon r^2}dr = \frac{Q}{4\pi\epsilon}\left[\frac{1}{r}\right]_{r_B}^{r_A} = \frac{Q}{4\pi\epsilon}\left[\frac{1}{r_A}-\frac{1}{r_B}\right] & (3.8)\end{aligned}$$

Then, if the reference point B is at infinity, we have the potential at any point A with radius vector $\mathbf{r}$:

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon r} \tag{3.9}$$

e.g.2 Potential near a line charge of density ρ_l

We know the field to be $\mathbf{E} = \frac{\rho_l}{2\pi\epsilon r}\hat{\mathbf{e}}_r$ in cylindrical co-ordinates, so $d\mathbf{l}$ is replaced by using

$$d\mathbf{l} = dr\,\hat{\mathbf{e}}_r + r\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z \tag{3.10}$$

The potential is then

$$\begin{aligned} V_{AB} &= -\int_B^A \mathbf{E}\cdot d\mathbf{l} = -\int_{r_B}^{r_A} E_r \cdot \hat{\mathbf{e}}_r\,dr = -\int_{r_B}^{r_A} E_r\,dr && (3.11) \\ &= -\int_{r_B}^{r_A} \frac{\rho_l}{2\pi\epsilon r}dr = \frac{\rho_l}{2\pi\epsilon}\left[\ln|r|\right]_{r_A}^{r_B} = \frac{\rho_l}{2\pi\epsilon}\left[\ln|r_B| - \ln|r_A|\right] && (3.12) \end{aligned}$$

With the reference of zero potential at $r_B = \infty$ we would get a nonsensical result, so we arbitrarily define zero potential as being at the fixed point $\mathbf{r}_o$. Hence, $\Phi(\mathbf{r}_o) = 0$ which gives us:

$$V_{OB} = \frac{\rho_l}{2\pi\epsilon}\left[\ln|r_B| - \ln|r_o|\right] = 0 \tag{3.13}$$

so

$$\ln|r_B| = \ln|r_o| \tag{3.14}$$

and so, in general:

$$\Phi(\mathbf{r}) = -\frac{\rho_l}{2\pi\epsilon}\left[\ln|r| - \ln|r_o|\right] = -\frac{\rho_l}{2\pi\epsilon}\left[\ln|r| + C\right] \tag{3.15}$$

where, again, C is just a constant.

$\oint D \; ds = \int \rho \, dv = Q$

e.g.3 Potential of an infinite sheet of density ρ_S

We first need an expression for the electric field near the sheet. One way to get this is to construct the cylindrical special gaussian surface shown in the diagram. Then, by Gauss's Law, the small enclosed charge q is given by $\oint_S \mathbf{D} \cdot d\mathbf{S} = q$. We can imagine the length of the cylinder to be arbitrarily short, so that flux flows through both flat ends of the gaussian cylinder. Thus:

$$q = \int_1 \mathbf{D}_1 \cdot d\mathbf{S}_1 + \int_2 \mathbf{D}_2 \cdot d\mathbf{S}_2 = Da + Da \tag{3.16}$$

Hence, $D = q/2a$.

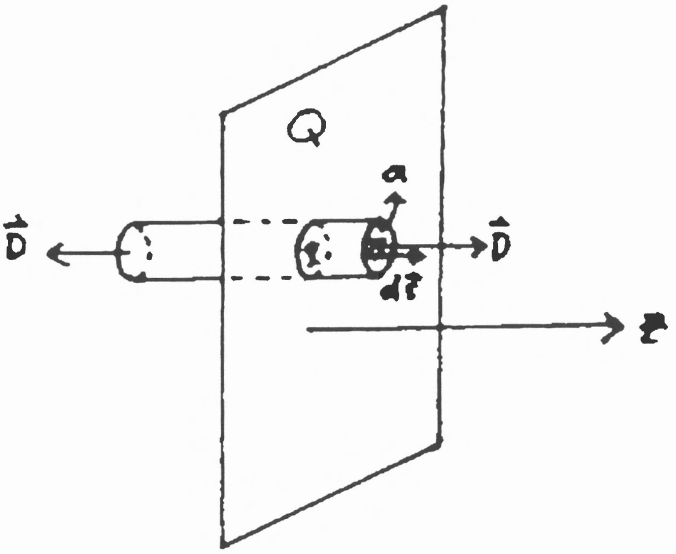
Since the position chosen for the gaussian cylinder was in no way special, and since $q/a = \rho_S$, we can write for the whole sheet that $D = \rho_S/2$ and so

$$\mathbf{E} = \pm\frac{\rho_S}{2\epsilon}\hat{\mathbf{e}}_z \tag{3.17}$$

with the E-field acting only in the $\pm\hat{\mathbf{z}}$-directions. Then, setting the reference potential at the sheet (at $z = 0$) as $\Phi(z_B) = V_o$, we can find the potential:

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{l} = -\int_{z_B}^{z_A} E_z \cdot \hat{\mathbf{e}}_z \, dz = \left[\frac{\rho_S}{2\epsilon}z\right]_{z_A}^{z_B} = V_o - \frac{\rho_S}{2\epsilon}z_A \tag{3.18}$$

at a perpendicular distance z_A from the sheet of charge in the positive $\hat{\mathbf{z}}$-direction.



3.3 The gradient of the potential

- In the vicinity of charges, we now know what is meant by the electric field and by the potential function. If you join points of equal potential in an electrostatic field, you get what are known as *equipotential lines.* We now demonstrate that these lines of equipotential are *always at right angles to the field lines.*

- Since $\Phi = -\int \mathbf{E} \cdot d\mathbf{l}$, $d\Phi = -\mathbf{E} \cdot d\mathbf{l}$. In cartesian co-ordinates, we can replace $d\mathbf{l}$ with $\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz$ and $\mathbf{E}$ with $E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ (note the slight change in notation), so

$$d\Phi = -E_x dx - E_y dy - E_z dz \tag{3.19}$$

Now Φ, too, is just a function of x, y and z, so we may write (see MAM280W):

$$d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz \tag{3.20}$$

from which, clearly

$$E_x = -\frac{\partial \Phi}{\partial x} \quad E_y = -\frac{\partial \Phi}{\partial y} \quad E_z = -\frac{\partial \Phi}{\partial z} \tag{3.21}$$

- Hence, we can write the field $\mathbf{E}$ in terms of the potential *only* by stating

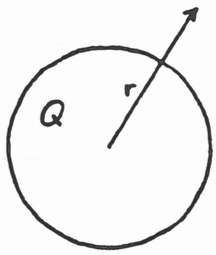
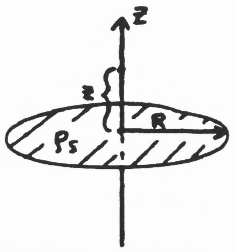
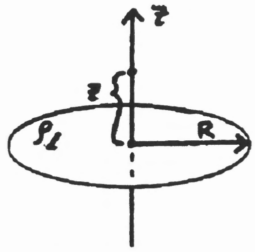
$$\mathbf{E} = -\left(\hat{\mathbf{x}}\frac{\partial \Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial \Phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial \Phi}{\partial z}\right) \triangleq -\nabla\Phi \tag{3.22}$$

where the vector operator ∇ should be very familiar to you!

- Now, since $\nabla\Phi$ is a vector showing the magnitude and direction of the *greatest* variation in potential seen from a point, the field lines lie in that direction too ($\mathbf{E} = -\nabla\Phi$). Equipotentials are the lines of *least* variation in potential, along which $d\Phi = \nabla\Phi \cdot d\mathbf{l} = 0$, and this condition will only apply if $\nabla\Phi \perp d\mathbf{l}$. Hence, *the equipotentials are everywhere perpendicular to the field lines*, which was what we set out to prove.

3.4 Standard charge configuration potentials

The table below gives some standard charge configurations of high symmetry for which you might like to calculate the potential functions. You will probably need to research the methods for doing these calculations, and should work in groups and look for texts that give guidance. Make sure that you are each able to perform these derivations independently, having done the research together.

Find the potential of:	Sketch of the configuration
1. A sphere of charge Q in a vacuum, at a distance r from the centre of the sphere where r is greater than the sphere's radius	
2. A circular disc of charge of radius R and charge density ρ_S, at a distance z along the z-axis which is the perpendicular axis of the disc	
3. A circular ring of charge of radius R and charge density ρ_l, at a distance z along the z-axis which is the perpendicular axis of the ring	

Chapter 4

The Field Between Electrodes

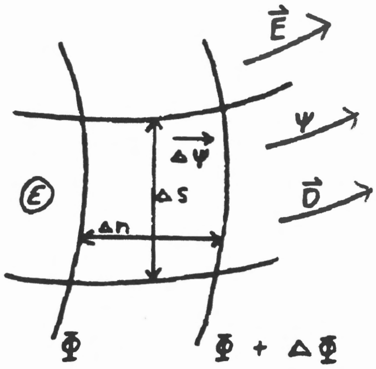
4.1 Graphical field mapping

- We now develop a technique for drawing the field and equipotential lines in the vicinity of charged electrodes of arbitrary shape. Our drawings will, of course, be 2-dimensional, but they are a good approximation for more realistic cases.

- We start with a known potential difference between two electrodes drawn in a diagram. Divide this difference into a convenient number of potential levels, and add to the diagram (using commonsense and some guesswork) an equipotential line for each level.

- It is best to start in regions where the field is straightforward and uniform. Thereafter, note that the equipotentials will tend to hug sharp, acute angles in the electrode boundaries and to avoid regions of reflex boundary curvature.

- Then draw in the field lines, perhaps using a different colour. Remember that the field lines are *everywhere orthogonal* to the equipotentials. Ensure that the resulting "curvilinear rectangles" are all similar in their side-size ratio. This will mean that you are drawing regularly-spaced lines of potential and field in your plot.



- The proof of the above statement is as follows. In the diagram of a single curvilinear rectangle in a plotted field, Δn is the distance between two adjacent equipotentials, and Δs separates two adjacent field lines. The field strength in this small region is therefore $\Delta\Phi/\Delta n$ V/m. Now the flux flowing between the field lines can be written as

$$\Delta\Psi = D\Delta s = \epsilon E\Delta s = \frac{\epsilon\Delta\Phi\Delta s}{\Delta n} \tag{4.1}$$

(note that we are so far ignoring the depth of the field into the paper).

- From this we have

$$\frac{\Delta s}{\Delta n} = \frac{\Delta\Psi}{\epsilon\Delta\Phi} \tag{4.2}$$

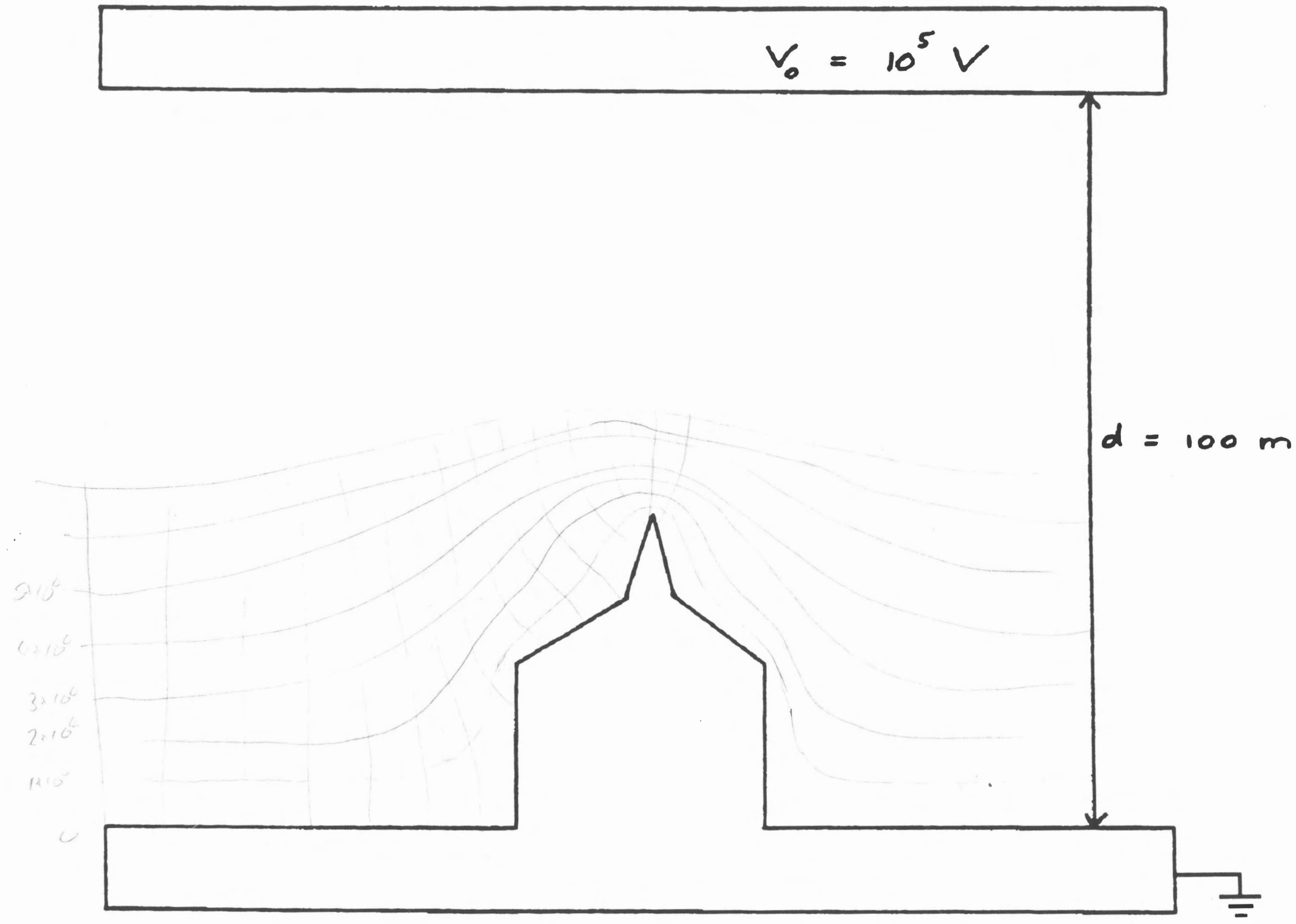
so if the flux per tube, $\Delta\Psi$, the potential difference per division, $\Delta\Phi$, and the permittivity, ϵ, are all constant, then the side ratio, $\Delta s/\Delta n$ will be constant too.

- If your rectangles are not all similar, then your first guesses about the positions of the equipotentials were at fault. Do not be afraid to redraw the diagram several times, until you are satisfied with the result.

4.2 Field strength

- You can read the field strength straight from a field map. For example, if V_o in the drawing on the next page is 100 kV and d = 100 m, then $E = 10^5/10^2 = 1000\,\mathrm{V/m}$ at each edge of the drawing. Above the spike, however, this value is much greater because of the tendency of the equipotentials to hug the sharp angle.

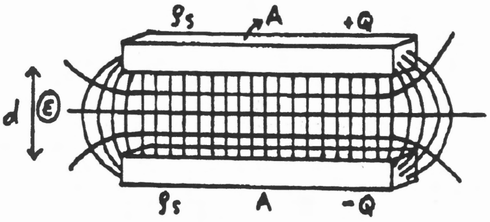
e.g. Draw equipotentials and electric field lines between these objects



4.3 Capacitance

- The *capacitance* between two electrodes of excess charge $\pm Q$ with a potential difference between them of $\Phi_A - \Phi_B$ is defined as

$$C = \frac{Q}{\Phi_A - \Phi_B} \tag{4.3}$$



- It is easy to sketch the equipotentials and field lines for a *parallel plate capacitor.* Ignoring for a moment the *fringing flux* that curves around the edge of the plates, we can obtain, from Gauss's Law, $D = \rho_S/2$ for each plate. So with both plates considered, $D = \rho_S$, the surface charge density. Then by definition of potential, and with the plates perpendicular to a cartesian z-axis:

$$\Phi_A - \Phi_B = -\int_B^A \mathbf{E}\cdot d\mathbf{l} = -\int_B^A \frac{D_z}{\epsilon}\cdot\hat{\mathbf{z}}\,dz = -\left[\frac{\rho_S z}{\epsilon}\right]_{z_B}^{z_A} = \frac{\rho_S d}{\epsilon} \tag{4.4}$$

- If each plate is of area A then evidently $Q = \rho_S A$, so we get the well-known equation for parallel-plate capacitance:

$$C = \frac{Q}{\Phi_A - \Phi_B} = \frac{\rho_S A}{\rho_S d/\epsilon} = \frac{\epsilon A}{d} \tag{4.5}$$

- Capacitance per unit length can be calculated from graphical field maps with two electrodes. The charge between the electrodes is the same as the flux ($\Psi = C$), and both are equal to the number of "flux tubes", N_f, multiplied by the flux per tube, $\Delta\Psi$. Similarly, the potential difference between the electrodes is just the number of potential divisions, N_p, times the potential difference per division, $\Delta\Phi$. Hence

$$C = \frac{Q}{\Phi_A - \Phi_B} = \frac{N_f \Delta\Psi}{N_p \Delta\Phi} = \frac{N_f}{N_p}\left(\frac{\epsilon\Delta s}{\Delta n}\right) \approx \epsilon\frac{N_f}{N_p} \tag{4.6}$$

assuming that the graphical field map consists of *small squares* so that $\Delta s \approx \Delta n$. Thus in the parallel-plate capacitor shown above, there are 4 potential divisions and 26 flux tubes, so the capacitance is

$$C \approx 10^{-9}/36\pi \ \times\ 26/4 \ = 57.5\text{pF/m}$$

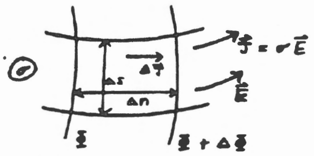
4.4 Conductance

- Up to now, we have assumed that the electrodes in our plots were surrounded by a vacuum ($\epsilon = \epsilon_o$) or by an insulating dielectric ($\epsilon = \epsilon_o \epsilon_r$). If, however, the surrounding medium is *conductive*, then the electric field **E** due to the charge on the electrodes will apply a force on the free electrons in the conducting medium. This will cause the electrons to move, and hence to establish a *current*. The *current density* (**J** in Amps/m^2) that results is related to the electric field by

$$\mathbf{J} = \sigma \mathbf{E} \tag{4.7}$$

where σ is the *conductivity* of the medium.

- It should be clear, considering a small region of the medium, that



$$\Delta J = J\Delta s = \sigma E \Delta s = \frac{\sigma \Delta\Phi \Delta s}{\Delta n} \tag{4.8}$$

so that

$$\frac{\Delta s}{\Delta n} = \frac{\Delta J}{\sigma \Delta\Phi} \tag{4.9}$$

- The *conductance* per unit length between the electrodes is defined as

$$G = \frac{I}{\Phi_A - \Phi_B} \quad \text{(in ℧/m)} \tag{4.10}$$

where the *total current*, I, is simply $N_f \Delta J$ (because the current flows along the flux lines). Hence

$$G = \frac{I}{\Phi_A - \Phi_B} = \frac{N_f \Delta J}{N_p \Delta\Phi} = \frac{N_f}{N_p}\left(\frac{\sigma\Delta\Phi\Delta s}{\Delta n \Delta\Phi}\right) \approx \sigma \frac{N_f}{N_p} \tag{4.11}$$

so long as the rectangles on the graphical field map are *small squares* for which $\Delta s / \Delta n \approx 1$

- We arrive finally at the useful relation

$$G = \frac{\sigma}{\epsilon} C \tag{4.12}$$

which you will meet in your work on transmission lines. Note that the *resistance* of a conductive medium is just $R = 1/G$ (in Ω). This gives us a way to find the resistance in a field map if σ is known.

Chapter 5

Divergence and Laplace's Equation

5.1 Integral and differential forms

- We have seen electric field intensity and electric potential related by equations in both integral and differential form. Thus

$$\Phi = -\int \mathbf{E} \cdot d\mathbf{l} \quad \text{and} \quad \mathbf{E} = -\nabla\Phi \tag{5.1}$$

- Gauss's Law in integral form is $\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$, where we recall that the charge, Q, may be written as $Q = \int_V \rho \, dV$. Differentiating both sides amounts to dividing through by ΔV and taking the limit as $\Delta V \rightarrow 0$:

$$\lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta V} = \lim_{\Delta v \rightarrow 0} \frac{\int_V \rho dV}{\Delta V} \tag{5.2}$$

  We recognise the RHS of the above as simply ρ, and we propose to write the LHS as $\nabla \cdot \mathbf{D}$, pronounced *div* **D**. Hence:

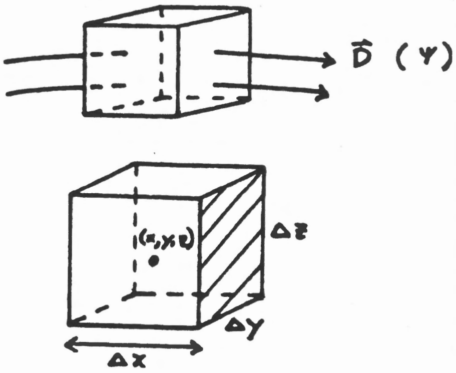
$$\nabla \cdot \mathbf{D} = \rho \tag{5.3}$$

  We call this the *differential form* of Gauss's Law.

5.2 The meaning of divergence

- Divergence refers to the *net* amount of a vector quantity per unit volume that emerges from an infinitesimal volume surrounding some point. It is therefore an expression of whether the quantity has a *source* at the point of interest. We can see what this means by considering a small cubic volume, ΔV, that lies within a field as shown.

- If the small cube lies inside an electrostatic field, it should be clear that the flux will be the same through any two of its opposite walls unless the flux density varies between two other walls, perpendicular to the first two walls. In fact, for a *small* volume, the difference in any vector between two opposite faces is approximately the gradient of the function times the distance between faces. This approximation becomes exact in the limit as the sides of our small cube tend to zero length.



- Let the small cube be centred at (x, y, z) and have dimensions $\Delta x \times \Delta y \times \Delta z$. Considering the x-component of $\mathbf{D}$, written D_x, the above theorem gives us expressions for the flux density in the x-direction at the sides of the cube:

$$\begin{aligned} D_x\left(x+\frac{\Delta x}{2}\right) &\approx D_x(x)+\frac{\Delta x}{2}\frac{\partial D_x(x)}{\partial x} \\ D_x\left(x-\frac{\Delta x}{2}\right) &\approx D_x(x)-\frac{\Delta x}{2}\frac{\partial D_x(x)}{\partial x} \end{aligned} \tag{5.4}$$

- The flux *out of* the right-hand face, from the first flux density, is:

$$\Delta y \Delta z D_x\left(x+\frac{\Delta x}{2}\right)=\Delta y \Delta z\left(D_x(x)+\frac{\Delta x}{2}\frac{\partial D_x(x)}{\partial x}\right) \tag{5.5}$$

  and the flux *into* the left hand face is:

$$\Delta y \Delta z D_x\left(x-\frac{\Delta x}{2}\right)=\Delta y \Delta z\left(D_x(x)-\frac{\Delta x}{2}\frac{\partial D_x(x)}{\partial x}\right) \tag{5.6}$$

Hence the *net flux out* of the cubic volume in the x-direction is the difference of these two:

$$\text{Net flux out (x direction)} = \Delta y \Delta z \Delta x \frac{\partial D_x}{\partial x} \tag{5.7}$$

You can then argue similarly for the y- and z- components of **D** to see that the total flux out of the small volume is:

$$\text{Total flux out} = \Delta x \Delta y \Delta z \frac{\partial D_x}{\partial x} + \Delta x \Delta y \Delta z \frac{\partial D_y}{\partial y} + \Delta x \Delta y \Delta z \frac{\partial D_z}{\partial z} \tag{5.8}$$

- Now, by Gauss's Law, the total flux out of our small volume must equal the charge that it encloses, which is just $\rho \Delta x \Delta y \Delta z$, so in the limit we can now write:

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \tag{5.9}$$

  But we can define *div* **D** as the vector operation:

$$div\ \mathbf{D} \equiv \nabla \cdot \mathbf{D} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \tag{5.10}$$

  and so we again conclude that $\nabla \cdot \mathbf{D} = \rho$

- Note the compactness of this elegant expression: it says that "the source of flux density is charge", something we began with back in lecture 1, but which we now see to be another aspect of Gauss's Law. Note also that you can get back to the integral form of Gauss's Law by using the *divergence theorem*, a mathematical identity that holds for any vector, **F**:

$$\int_V \nabla \cdot \mathbf{F}\, dV = \oint_S \mathbf{F} \cdot d\mathbf{S} \tag{5.11}$$

  Integrating $\rho = \nabla \cdot \mathbf{D}$ and using this theorem, we get

$$\int_V \rho\, dV = \int_V \nabla \cdot \mathbf{D}\, dV = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q \tag{5.12}$$

Repeat all this for cylindrical and spherical coordinates.

- $\nabla \cdot \mathbf{D}$ can also be defined in cylindrical and in spherical co-ordinates, using arguments much like the one presented above for the cartesian case. You should make sure that you have researched these arguments, and are capable of showing the following from first principles:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} \quad \text{(sph.)} \\ \nabla \cdot \mathbf{D} &= \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z} \quad \text{(cyl.)} \end{aligned} \tag{5.13}$$

e.g.1 Find $\nabla \cdot \mathbf{D}$ for a solid sphere of charge density ρ and radius a, embedded in a vacuum

Inside the sphere, using a gaussian sphere of radius $r < a$, Gauss's Law gives

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \quad \text{so} \quad D.4\pi r^2 = \frac{4}{3}\pi r^3 \rho \quad \text{so} \quad D = \frac{r\rho}{3} \quad \text{so} \quad \mathbf{D} = \frac{r\rho}{3}\hat{\mathbf{e}}_r \tag{5.14}$$

Then, using spherical co-ordinates,

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{r\rho}{3}\right) = \frac{1}{r^2}\frac{3r^2\rho}{3} = \rho \tag{5.15}$$

exactly as expected in a region of charge density ρ.

Outside the sphere we work with a gaussian sphere of radius $r > a$ to get

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \quad \text{so} \quad D.4\pi r^2 = \frac{4}{3}\pi a^3 \rho \quad \text{so} \quad D = \frac{a^3\rho}{3r^2} \quad \text{so} \quad \mathbf{D} = \frac{a^3\rho}{3r^2}\hat{\mathbf{e}}_r \tag{5.16}$$

Then, using spherical co-ordinates,

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{a^3\rho}{3r^2}\right) = 0 \tag{5.17}$$

again as expected in a region of zero charge density.

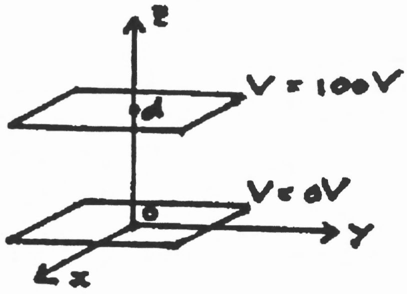
5.3 The equations of Poisson and Laplace

- Using the three relations $\nabla \cdot \mathbf{D} = \rho$, $\mathbf{E} = -\nabla\Phi$ and $\mathbf{D} = \epsilon\mathbf{E}$, we can now write
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} = -\nabla \cdot \nabla\Phi \tag{5.18}$$
and this is generally seen as a differential equation called *Poisson's equation*, which is given below in several forms:
$$\nabla \cdot \nabla\Phi \equiv \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = -\frac{\rho}{\epsilon} \tag{5.19}$$
where $\nabla^2\Phi = div(grad\ \Phi)$ is called the *Laplacian* of Φ.

- In regions of no charge, another important equation, called *Laplace's equation*, results:
$$\nabla^2\Phi = 0 \tag{5.20}$$
Solving Laplace's equation to find the potential function Φ amounts to solving a second order, partial differential equation in which Φ may vary in 1, 2 or 3 dimensions, and may be expressed in cartesian, cylindrical or spherical co-ordinates. As you can imagine, the mathematics can vary from very easy through to rather complicated! We restrict ourselves to some very simple examples below.

- In order that you can be comfortable with Laplace's equation in cylindrical and spherical co-ordinates, you should, in your own time, take the expressions that you derived for $\nabla \cdot \mathbf{D}$ in those co-ordinate systems and show that Laplace's equation becomes
$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \quad \text{(sph.)} \\
\nabla^2\Phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} \quad \text{(cyl.)} \qquad (5.21)
\end{aligned}$$

e.g.2 Potential between parallel conducting plates



If the region between the parallel plates is charge-free, then we can write:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{5.22}$$

Neglecting fringing, the potential is a function of z only, so $\frac{d^2\Phi}{dz^2} = 0$, from which we can integrate once to get $\frac{d\Phi}{dz} = A$ and integrate again to obtain

$$\Phi = Az + B \tag{5.23}$$

We can evaluate the constants A and B by using the known values of the potential given to us as the voltages of the plates. Using this information is known as *applying the boundary conditions.*

So, since at $z = 0$ we know $\Phi = 0$, we can see that $B = 0$
and since at $z = d$ we know $\Phi = 100$, we also obtain $A = 100/d$. Thus

$$\Phi = 100\left(\frac{z}{d}\right) \quad \text{(volts)} \tag{5.24}$$

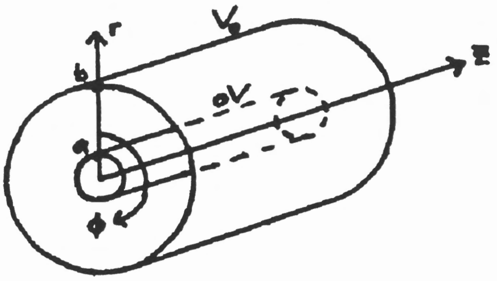
This gives us an expression for the potential at any point (x, y, z) between the plates, and it suggests that equipotential levels would be equally-spaced, which we know to be the case. We can now find the expression for the electric field intensity, like this:

$$\mathbf{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} - \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} - \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}} = -\frac{d}{dz}\left(100\frac{z}{d}\right)\hat{\mathbf{z}} = -\frac{100}{d}\hat{\mathbf{z}} \tag{5.25}$$

and you should recognise this as having exactly the magnitude and direction that we would expect for an ideal, parallel-plate capacitor

e.g.3 Potential between coaxial cylinders

If the region between the coaxial cylinders is charge-free, then we can write Laplace's equation in cylindrical co-ordinates:



$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{5.26}$$

Assuming no variation in potential with changing angle ϕ, and also assuming that the cylinder is long enough that there is no variation of potential with z, Laplace's equation reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = 0 \quad \text{so} \quad \frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = 0 \quad \text{so} \quad r\frac{d\Phi}{dr} = A$$

$$\text{so} \quad \frac{d\Phi}{dr} = A/r \quad \text{so} \quad \Phi = A\ln|r| + B \tag{5.27}$$

Next we apply the boundary conditions:

Since at $r = a$ we know $\Phi = 0$, we can see that $A\ln|a| + B = 0$ and

since at $r = b$ we know $\Phi = V_o$, we also get $A\ln|b| + B = V_o$,

and we can then easily obtain

$$A = \frac{V_o}{\ln|b| - \ln|a|} \quad \text{and} \quad B = \frac{-V_o\ln|a|}{(\ln|b| - \ln|a|)} \tag{5.28}$$

The expression $\Phi = A\ln|r| + B$ can now be used to find the potential at any radius r, knowing only the dimensions a and b of the coaxial cylinders, and the voltage on the outer cylinder with respect to the inner.

Charge on hit: Q3.a occupies xz plane.

Chapter 6

The Uniqueness Theorem and Energy

6.1 The uniqueness theorem

- There is an important theorem which states that if we are given a region where Laplace's equation holds, and if we know the boundary conditions, then the solution to the equation is the *unique* expression of the potential of the field in that region.

- We can prove this *uniqueness theorem* by contradiction. Imagine that we have a charge-free region, with potentials specified on its boundary. Let us also imagine that there are *two* solutions for this potential, written Φ_1 and Φ_2. At the boundary, both solutions must give the same potential (the potential specified), and so we can write *for the boundary*:

$$\Phi_1 = \Phi_2 \quad \text{so} \quad \Phi_1 - \Phi_2 = 0 \tag{6.1}$$

- Both potential functions would have to be solutions of Laplace's equation, and so we can write *for the whole region*:

$$\nabla^2\Phi_1 = 0 \quad \text{and} \quad \nabla^2\Phi_2 = 0 \quad \text{and hence} \quad \nabla^2(\Phi_1 - \Phi_2) = 0 \tag{6.2}$$

- Now recall the *divergence theorem*: for any vector **F**

$$\int_V \nabla \cdot \mathbf{F} dV = \oint \mathbf{F} \cdot d\mathbf{S} \tag{6.3}$$

We choose to use the divergence theorem with $\mathbf{F} = (\Phi_1 - \Phi_2)\nabla(\Phi_1 - \Phi_2)$. This gives

Ψ $\vec{A}$

$$\int_V \nabla \cdot [(\Phi_1 - \Phi_2)\nabla(\Phi_1 - \Phi_2)] dV = \oint [(\Phi_1 - \Phi_2)\nabla(\Phi_1 - \Phi_2)] \cdot d\mathbf{S} \tag{6.4}$$

- Another mathematical identity (which you may find it very instructive to prove to yourself) states that for a vector **A** and a scalar Ψ,

~~Proo~~ prove this

$$\nabla \cdot (\Psi \mathbf{A}) = \Psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \Psi \tag{6.5}$$

$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$ $\Psi = \Psi(x,y,z)$

Noting that $\Phi_1 - \Phi_2$ is a scalar and $\nabla(\Phi_1 - \Phi_2)$ is a vector, the previous equation may now be written

$\Psi \nabla . \vec{A} =$

$\Psi \frac{\partial A}{\partial x} + \Psi \frac{\partial A}{\partial y} + \Psi \frac{\partial A}{\partial z}$

on surface, at boundary conditions, $\Phi_1 - \Phi_2 = 0$

$$\int_V (\Phi_1 - \Phi_2)\nabla^2(\Phi_1 - \Phi_2) dV + \int_V [\nabla(\Phi_1 - \Phi_2)]^2 dV = \oint_S (\Phi_1 - \Phi_2)\nabla(\Phi_1 - \Phi_2) \cdot d\mathbf{S} \tag{6.6}$$

$\nabla \Psi =$

$\cdot \nabla \Psi = A_x \frac{\partial \Psi}{\partial x} + A_y \frac{\partial \Psi}{\partial y} +$

- Now the first term of equation 6.6 refers to the whole region and equals zero (by equation 6.2), and the term on the RHS refers to the boundary surface, and equals zero (by equation 6.1). We are left with

$$\int_V [\nabla(\Phi_1 - \Phi_2)]^2 dV = 0 \tag{6.7}$$

The gradient of a real scalar being real, and therefore having a square which is positive or zero, the gradient $\nabla(\Phi_1 - \Phi_2)$ can only be zero. Integrating, we obtain a relationship between the two potential functions which must apply everywhere:

$$\Phi_1 - \Phi_2 = \text{constant} \tag{6.8}$$

- But if this applies everywhere, then it applies at the boundary, where we know the two potentials are equal. We are obliged to conclude that the constant in the equation above is zero, and that the two potential functions are equal. Our initial assumption of two different potential solutions has led to a contradiction, and we have proved that *solutions to Laplace's equation with boundary conditions given are unique.*

6.2 Energy in fields

- Evidently, electric fields store energy (they are capable of accelerating charge q through a potential drop v, and the energy required to do this is $w = qv$). We now aim to find expressions for the electrostatic energy stored in a field, in terms of field quantities such as $\mathbf{E}$, Φ and $\mathbf{D}$.

- The energy required to assemble a group of n charges, bringing each charge from zero potential at infinite distance to a particular potential, Φ_i, in an arrangement or system of charges is

  $$U = \frac{1}{2}\sum_{i=1}^{n} q_i\Phi_i \tag{6.9}$$

  where the factor of $1/2$ compensates for a formula that would otherwise count the contribution of each charge *twice*: once as it is brought up to the other charges, and once as each of them is brought towards it.

- If the charge distribution is continuous, with charge density ρ, we write

  $$U = \frac{1}{2}\int_V \rho\Phi\, dV \tag{6.10}$$

  where the potential, Φ, is a function of the position within the volume over which the integral is performed. Recalling that $\rho = \nabla \cdot \mathbf{D}$, this becomes

  $$U = \frac{1}{2}\int_V (\nabla \cdot \mathbf{D})\Phi\, dV \tag{6.11}$$

and then, since for any scalar ψ and vector $\mathbf{A}$ we have

$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi \quad \text{so} \quad \psi(\nabla\cdot\mathbf{A}) = \nabla\cdot(\psi\mathbf{A}) - \mathbf{A}\cdot\nabla\psi \tag{6.12}$$

(you have recently checked the proof of this for yourself), we can write

$$U = \frac{1}{2}\int_V \nabla\cdot(\Phi\mathbf{D})\,dV - \frac{1}{2}\int_V \mathbf{D}\cdot(\nabla\Phi)\,dV \tag{6.13}$$

- The first term of this, by the divergence theorem, now reduces to

drops like $\frac{1}{r^2}$ drops like $\frac{1}{r}$ area rises like r^2

$$\frac{1}{2}\int_V \nabla\cdot(\Phi\mathbf{D})\,dV = \frac{1}{2}\oint_S \Phi\mathbf{D}\cdot d\mathbf{S} = 0 \tag{6.14}$$

take this to infinity drops off to zero.

since we are interested in the total energy due to every part of all of the fields of the charges present, and this involves integrating over a surface at infinity, at which distance the product of $\mathbf{D}$ and Φ will have shrunk to nothing compared with the area of the surface of integration.

- We are therefore left with

$$U = -\frac{1}{2}\int_V \mathbf{D}\cdot(\nabla\Phi)\,dV = \frac{1}{2}\int_V \mathbf{D}\cdot\mathbf{E}\,dV \tag{6.15}$$

This equation confirms that total energy resides in the field represented by $\mathbf{D}$ or $\mathbf{E}$, and we can use it to express the energy present in each small volume element of the region by differentiating to get

$$dU = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dV \tag{6.16}$$

- Considering now a parallel-plate capacitor, in which
  - the plates have area A and separation d;
  - the voltage drop between the plates is V_c;
  - the electric field has constant value $E = V_c/d$;
  - flux density $D = \epsilon E = \epsilon V_c/d$ is constant, as ϵ is constant:

  the energy stored in the capacitor's field (neglecting fringing) is

$$\begin{aligned} U_c &= \frac{1}{2} DE \int_V dV && (6.17) \\ &= \frac{1}{2}\left(\frac{\epsilon V_c}{d}\right)\left(\frac{V_c}{d}\right)(Ad) && (6.18) \\ &= \frac{1}{2}\left(\frac{\epsilon A}{d}\right) V_c^2 && (6.19) \end{aligned}$$

  But we recognise from equation 4.5 that $\epsilon A/d = C$, the capacitance of the parallel-plate system; and so we have, just from field theory considerations, the well-known result

$$U_c = \frac{1}{2} C V_c^2 \qquad (6.20)$$

Chapter 7

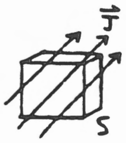
Electrostatic Boundary Conditions

7.1 Electric field due to dc current

- When a dc potential is applied to a conductor, a steady dc current will flow. The electrons that participate in the current give rise to an electric field which, although not strictly static, is of interest to us now because it does not vary in time. Such fields are called *quasistatic* fields, and analysis of them proceeds similarly to our study of electrostatic fields.

- In an *ohmic* conductor (a conductor obeying Ohm's Law), the *current density*, **J**, is related to the applied field **E** via the *conductivity*, σ. Furthermore, the relationship between electric field and potential allows us to write

$$\mathbf{J} = \sigma\mathbf{E} = -\sigma\nabla\Phi \tag{7.1}$$



- A *stationary* (i.e. non-time-varying) current will remove exactly as much charge from any closed region as it adds to it, per unit of time. There can be no build-up of charge either spontaneously, or due to a stationary current. This *principle of continuity* can be written

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = 0 \tag{7.2}$$

  or, dividing through by ΔV and taking the limit as $\Delta V \to 0$,

$$\nabla \cdot \mathbf{J} = 0 \tag{7.3}$$

- We therefore have $\nabla \cdot (-\sigma \nabla \Phi) = 0$, so with σ a constant we can write

$$\nabla \cdot \nabla \Phi \equiv \nabla^2 \Phi = 0 \tag{7.4}$$

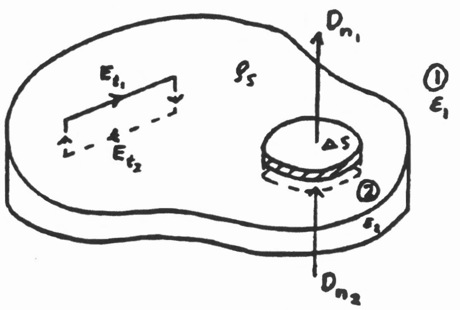
  In other words, the potential that is due to a stationary current satisfies Laplace's equation, just like the potential due to static charge.

- We have already seen that solving Laplace's equation amounts to solving a differential equation in several variables. The rest of this chapter investigates the *boundary conditions* that we use to evaluate the arbitrary constants in such calculations.

7.2 The boundary between dielectrics

- Since the field that we wish to study is often confined to a region occupied by one material and surrounded by other, different materials, we often gain useful information from knowing how **E**, **D**, Φ and **J** will behave at the boundary.

- Consider an arbitrary boundary between dielectrics 1 and 2 of permittivity ϵ_1 and ϵ_2, with a surface charge ρ_S on the boundary. We already know that the normal flux density, D_n, is the same in both materials if there is no surface charge. Thus, if $\rho_S = 0$, the component of **D** normal to the boundary is equal on both sides of it, and we can write



$$D_{n1} = D_{n2} \tag{7.5}$$

  This is not the case, however, if a surface charge is present.

- Imagine a cylindrical gaussian "pillbox" embedded in the two materials at the boundary as shown above. Let the area of each flat circular face of the cylinder be ΔS, and let the cylinder be so short that the tangential component of flux density passing through it is negligible.

The charge enclosed by the cylinder is $\rho_S \Delta S$, so clearly Gauss's Law gives

$$D_{n1}\Delta S - D_{n2}\Delta S = \rho_S \Delta S \tag{7.6}$$

from which

$$D_{n1} - D_{n2} = \rho_S \tag{7.7}$$

- If $\rho_S = 0$, then we can use $\mathbf{D} = \epsilon\mathbf{E}$ to see that

$$\epsilon_1 E_{n1} = \epsilon_2 E_{n2} \tag{7.8}$$

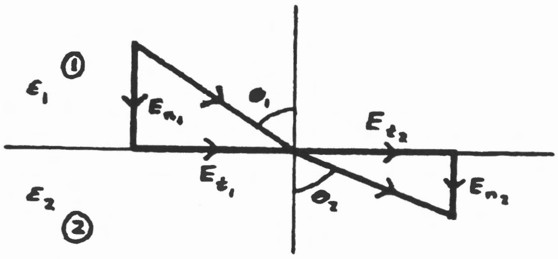
and that therefore the normal component of electric field intensity changes at the boundary because of the dielectric discontinuity. Now imagine traversing the closed path that crosses the dielectric boundary in the diagram. Let the long sides of the rectangular path have length Δl, and let the short sides be so short that their contributions to $\oint \mathbf{E}\cdot d\mathbf{l}$ as we traverse the path are negligible. This means in effect that the potentials on either side of the boundary are equal (i.e. $\Phi_1 = \Phi_2$). We know from the conservative property of the field that

$$\oint \mathbf{E}\cdot d\mathbf{l} = E_{t1}\Delta l - E_{t2}\Delta l = 0 \tag{7.9}$$

from which

$$E_{t1} = E_{t2} \quad \text{so} \quad \frac{D_{t1}}{\epsilon_1} = \frac{D_{t2}}{\epsilon_2} \tag{7.10}$$

- This vector diagram summarises the effect on $\mathbf{E}$ of a change in dielectric at the boundary.



It should be clear that

$$\theta_1 = \tan^{-1}\left(\frac{E_{t1}}{E_{n1}}\right) \quad \text{and} \quad \theta_2 = \tan^{-1}\left(\frac{E_{t2}}{E_{n2}}\right) \tag{7.11}$$

so, by equations 7.8 and 7.10,

$$\theta_2 = \tan^{-1}\left(\frac{E_{t1}}{E_{n1}}\frac{\epsilon_2}{\epsilon_1}\right) = \tan^{-1}\left(\frac{\epsilon_2}{\epsilon_1}\tan\theta_1\right) \tag{7.12}$$

and so if $\epsilon_2 > \epsilon_1$, then the total $\mathbf{E}$-vector is deflected *away from* the normal at the boundary.

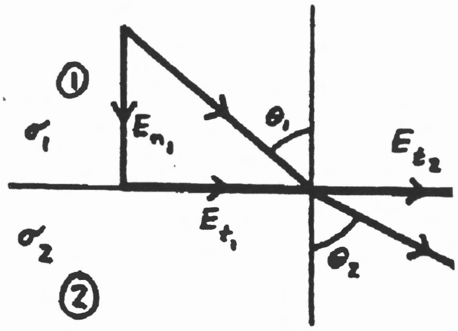
7.3 The boundary between conductors

- At the boundary between conductors, the normal component of current density, J_n, is unchanged across the boundary: this is a necessary continuity, because there would otherwise be a build-up of charge at the boundary. Hence

$$J_{n1} = J_{n2} \quad \text{so} \quad \sigma_1 E_{n1} = \sigma_2 E_{n2} \quad \text{and} \quad E_{n1} = \frac{\sigma_2}{\sigma_1} E_{n2} \tag{7.13}$$

- The tangential electric field is unchanged across the boundary for exactly the reason that we saw when considering two dielectrics. Hence $E_{t1} = E_{t2}$. Again, the angles θ_1 and θ_2 can be determined from the conductivities of the materials at the boundary, and equations very similar to equations 7.11 and 7.12 will be obtained. In particular,

$$\theta_2 = \tan^{-1}\left(\frac{\sigma_2}{\sigma_1}\tan\theta_1\right) \tag{7.14}$$

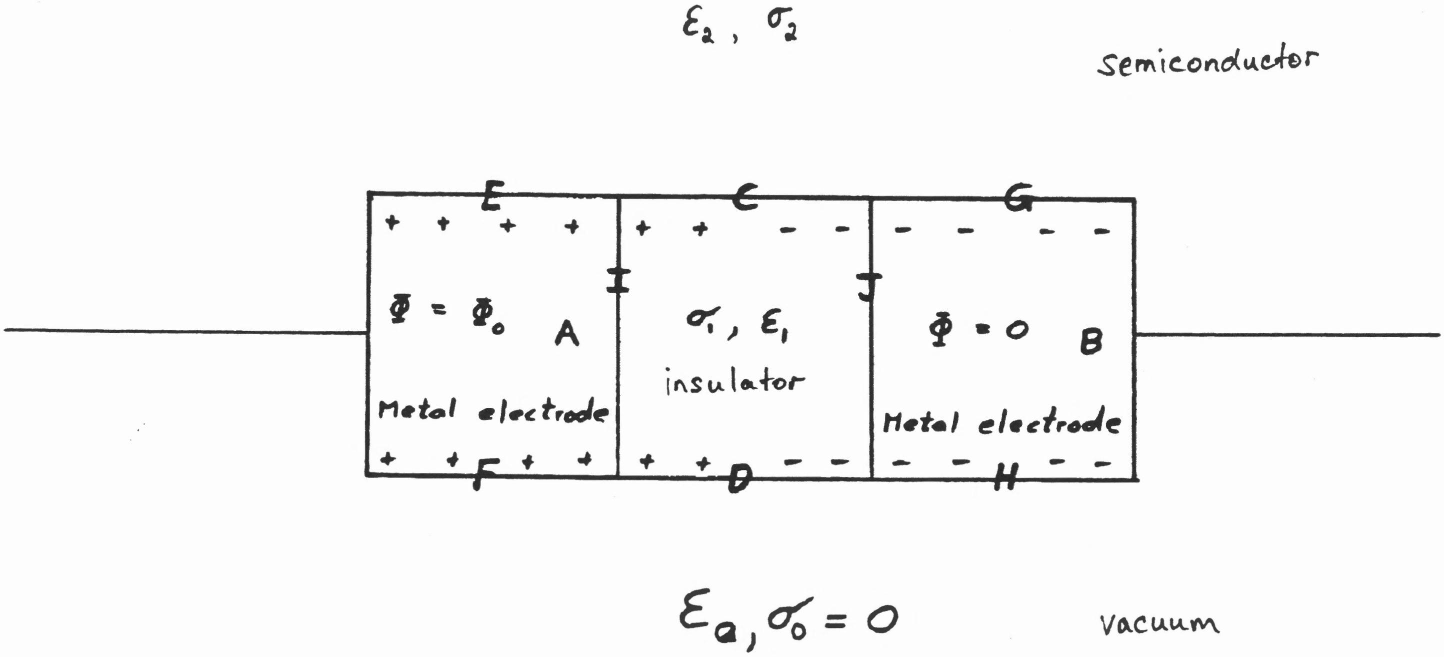


Note that if material 1 were a semiconductor and material 2 a good conductor such as a metal, then $\sigma_1 << \sigma_2$, so $\frac{\sigma_2}{\sigma_1} \rightarrow \infty$, so $\theta_2 \rightarrow 90^o$. We generally assume that metals are perfect conductors, in which both components of the electric field are zero.

- At the boundary between a conductor and an *insulator* ($\sigma = 0$), the normal component of current density, J_n, must be zero on both sides of the boundary. Between a *perfect* conductor and an insulator, the tangential component of electric field, E_t, must be zero on both sides of the boundary. For this reason, field lines *always leave or meet a perfect conductor at right angles.* Within a perfect conductor there can be no gradient of potential and therefore no electric field.

e.g. Example to illustrate boundary condition rules

List the boundary conditions and draw the field lines for the structure below. You can assume that $\sigma_2 >> \sigma_1$ and $\epsilon_1 > \epsilon_o$, with $\sigma_o = 0$.



Part 2

Magnetostatics

Chapter 8

Magnetostatic Fields

8.1 Introduction

- Magnets (lodestone) have been known for at least 800 years, and their use has had a profound effect on human history.
- Just as electric charges are either "positive" or "negative", magnetic "charges" or *poles* are of two types, called *north* and *south*. Note that magnetic poles are always only found in north-south pairs, never singly.
- Several basic observations about the behaviour of magnets have been made, and must be included in a full theory of how magnets work:
    - Magnetic material attracts iron (known from antiquity);
    - Magnets have poles of two types: like poles repel each other and unlike poles attract each other (from about 1600);
    - A magnetised compass needle is deflected by a nearby electric current (A heuristic discovery by Oersted in 1820);
    - A constant current can give rise to an *electromagnet* with very similar properties to a permanent magnet (also 1820);
    - Magnets will deflect an electric current flowing in a vacuum or semiconductor by the vacuum tube or Hall effect (20th century).
- We can explain these phenomena with the concept of *magnetic fields*.

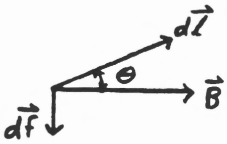
8.2 Static magnetic field

- A permanent magnet or a steady current is associated with a *magnetic field.* This is justified because the magnet or electromagnet is seen to exert a *force* on a nearby current-carrying wire, for instance. We know from electrostatics that forces in several dimensions are well-handled by fields.

- Whatever the source of the field (permanent magnet or electromagnet), we define a vector $\mathbf{B}$, called the *magnetic flux density*, so that the force exerted by the field on a small current element of length $d\mathbf{l}$ and carrying current I is of magnitude

$$df = I|d\mathbf{l}||\mathbf{B}|\sin\theta \tag{8.1}$$

where θ is the angle between $d\mathbf{l}$ and $\mathbf{B}$.



- Now $d\mathbf{f}$ is at right angles to both $d\mathbf{l}$ and $\mathbf{B}$, and determined by the advance of a right-hand screw when $d\mathbf{l}$ is rotated onto $\mathbf{B}$ through the smaller angle between them. This is conveniently written as

$$d\mathbf{f} = I\,d\mathbf{l} \times \mathbf{B} \tag{8.2}$$

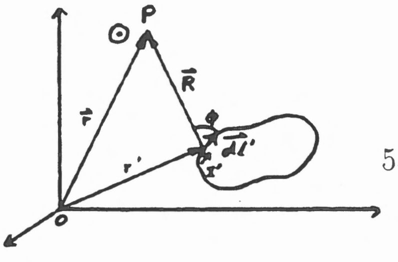
(Recall that if $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, then $\mathbf{C}$ is a vector of magnitude $|\mathbf{A}||\mathbf{B}|\sin\theta$, where θ is the smaller angle between $\mathbf{A}$ and $\mathbf{B}$. The direction of $\mathbf{C}$ is determined by the advance of a right-hand screw rotating $\mathbf{A}$ onto $\mathbf{B}$ through θ.)

- The magnetic flux density, $\mathbf{B}$, is measured in *tesla* (T) or *webers per square metre* ($\mathrm{Wbm^{-2}}$) or even $\mathrm{Vsm^{-2}}$. We also have use for another field quantity called the *magnetic field intensity*, $\mathbf{H}$, which is related to $\mathbf{B}$ via a property of the medium called the *permeability*, μ. Thus

$$\mathbf{B} = \mu\mathbf{H} \tag{8.3}$$

and $\mathbf{H}$ is measured in $\mathrm{Am^{-1}}$ while μ is in *henries per metre*, $\mathrm{Hm^{-1}}$. The permeability of free space is $\mu_o = 4\pi \times 10^{-7}\ \mathrm{Hm^{-1}}$.

8.3 The Biot-Savart Law



- Immediately following Oersted's discovery of the magnetic effect of a steady current, Biot, Savart and Ampère deduced the relationship that expresses the size and direction of the magnetic field intensity.

- The effect of a current I' flowing through a small line element of length $d\mathbf{l}'$ at a point in space defined as $\mathbf{r}'$ with respect to an arbitrary origin can be stated at any other point, P. If P is at the point defined by $\mathbf{r}$ from the origin, then the magnetic field intensity at P due to the flow of I' in $d\mathbf{l}'$ is given in magnitude by:

$$dH(\mathbf{r}) = \frac{I'(\mathbf{r}')d\mathbf{l}' \sin\phi}{4\pi R^2} \tag{8.4}$$

  where $R = |\mathbf{r} - \mathbf{r}'|$ and ϕ is the smaller angle between $\mathbf{R}$ and $d\mathbf{l}'$.

- As for the direction of $\mathbf{H}$, Biot and Savart observed this to be at right angles to both $d\mathbf{l}'$ and $\mathbf{R}$ and determined by the advance of a right-hand screw that rotates $d\mathbf{l}'$ onto $\mathbf{R}$ through the smaller of the two possible angles between them. From the definition of the cross product, we can therefore write the Biot-Savart law as

$$d\mathbf{H}(\mathbf{r}) = \frac{I'(\mathbf{r}')d\mathbf{l}' \times \mathbf{R}}{4\pi R^3} \tag{8.5}$$

  Note that in the example of the diagram above, $d\mathbf{H}$ at P is directed outward from the page. You can always quickly check the direction of magnetic field from the following "right-hand rule". If you imagine gripping the conductor with your thumb in the direction of the current, your fingers will all point in the direction of the magnetic field.

- The total magnetic flux intensity at P, arising from all of the current elements along a path (usually a closed path) is therefore

$$\mathbf{H}(\mathbf{r}) = \oint \frac{I'(\mathbf{r}')d\mathbf{l}' \times \mathbf{R}}{4\pi R^3} = \oint \frac{I'(\mathbf{r}')d\mathbf{l}' \times \hat{\mathbf{e}}_R}{4\pi R^2} \tag{8.6}$$

  where $\hat{\mathbf{e}}_R$ denotes the unit vector *from* each current element *to* the point, P, at which $\mathbf{H}$ is to be determined.

why now only 2

8.4 Magnetization and magnetic susceptibility

- You should notice the following correspondences, that will make our later work with static magnetic fields simpler, using the experience that we have already gained in electrostatics:

$$\mathbf{H} \leftrightarrow \mathbf{E} \tag{8.7}$$

$$\mathbf{B} \leftrightarrow \mathbf{D} \tag{8.8}$$

$$\mu \leftrightarrow \epsilon \tag{8.9}$$

- The medium surrounding the current does not affect the magnetic field intensity, **H**, but it appears in expressions for the magnetic flux density, **B**. Contrast this with the electrostatic case. The reason for this is that the medium consists of atoms whose electrons orbit and spin within their shells: these movements of the electrons can be regarded as currents in their own right, and such currents will produce a field of their own. This field is known as the *magnetization*, **M**, and it adds to the effect of the **H**-field that is due to the main current of interest.

- In a similar development to what we saw with *electric polarisation*, we can now relate **B** and **H** as if there were only free space in the region of interest, but with the magnetization adding to the **H**-field:

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}) \tag{8.10}$$

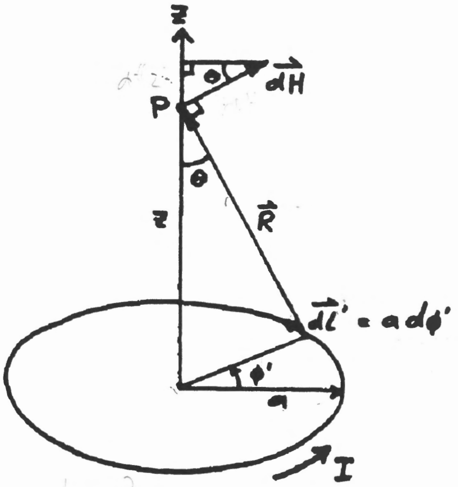
  We can then write

$$\mathbf{B} = \mu\mathbf{H} = \mu_o\mu_r\mathbf{H} = \mu_o(1 + \chi_m)\mathbf{H} \tag{8.11}$$

  where χ_m is the *magnetic susceptibility* of the material of the medium, and the *relative permeability* of the medium, μ_r, is merely $1 + \chi_m$.

- We will not deal further here with the physical causes of these magnetic effects in materials. You should, however, try to read something further in this area, because of its relevance to such interesting topics as data storage, superconductivity, plasma containment, NMR, etc.

e.g.1 Magnetic field on the axis of a circular current loop



- Let the loop carry current I and have the z-axis as its axis. Its radius is a, and we consider the magnetic field at an arbitrary point P on the axis. Let P be distance z from the centre of the loop.
- In cylindrical co-ordinates, a current element on the loop has length $dl' = ad\phi'$ and is always at right angles to the vector $\mathbf{R}$ that joins $d\mathbf{l}'$ to P. By Pythagoras' Theorem, $\mathbf{R}$ has length $\sqrt{a^2+z^2}$. Hence for each element of current, the magnitude of the field intensity is

ask - why not a^2

$$dH = \frac{Iad\phi' R\sin(90^o)}{4\pi R^3} = \frac{Iad\phi'}{4\pi(a^2+z^2)} \tag{8.12}$$

- Integrating around the loop, all of the radial contributions to $\mathbf{H}$ cancel, so only the z-components are relevant. We have

$\sin\theta = \frac{a}{R}$

$$dH_z = dH\sin\theta = \frac{adH}{\sqrt{(a^2+z^2)}} = \frac{Ia^2d\phi'}{4\pi(a^2+z^2)^{\frac{3}{2}}} \tag{8.13}$$

- To integrate for all values of ϕ', we just multiply by 2π, yielding

$$H_z = \frac{Ia^2}{2(a^2+z^2)^{\frac{3}{2}}} \quad \text{or} \quad \mathbf{H} = \frac{Ia^2}{2(a^2+z^2)^{\frac{3}{2}}}\hat{\mathbf{z}} \tag{8.14}$$

Note that in the centre of the circular loop (where $z = 0$), the H-field therefore has magnitude $I/2a$, and is entirely in the z-direction. This is in accordance with our expectation from the "right-hand rule"

e.g.2 Magnetic field of a straight line of current

- The geometry suggests the use of cylindrical co-ordinates.
- We consider a point P, distance r from the centre of the straight line. With I flowing as shown below, $\mathbf{H}$ will come out of the page by the "right-hand rule". There is no z-component of $\mathbf{H}$ since P is opposite the mid-point of the line, with the result that all z-components cancel. $\mathbf{H}$ is perpendicular to the page, and therefore has no r-component either. Hence, by equations 8.4 and 8.6,

$$H_\phi = \int_{-a}^{a} \frac{I \sin\phi \, dz}{4\pi R^2} \tag{8.15}$$

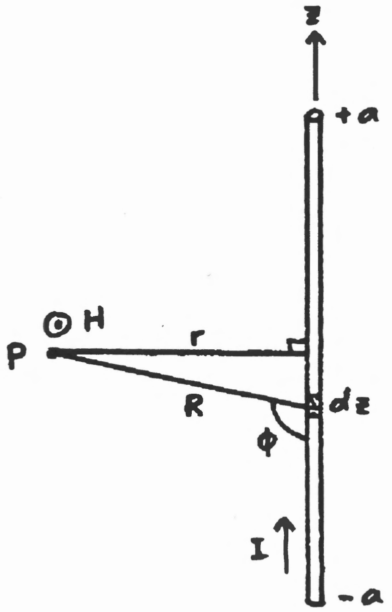
- Then, since $\sin\phi = r/R$ and $R = \sqrt{r^2+z^2}$, this gives

$$H_\phi = \frac{Ir}{4\pi}\int_{-a}^{a} \frac{dz}{(r^2+z^2)^{\frac{3}{2}}} = \frac{I}{2\pi r}\frac{1}{\sqrt{(r/a)^2+1}} \tag{8.16}$$

  from integration tables.

- This is the expression for the magnetic field of a *finite* line of current. In the case of an infinite line of current, $a \to \infty$, and then H_ϕ becomes

$$H_{\phi(\infty)} = \frac{I}{2\pi r} \quad \text{or} \quad \mathbf{H} = \frac{I}{2\pi r}\hat{\phi} \tag{8.17}$$



Chapter 9

Ampère's Circuital Law

All [illegible] place [illegible]

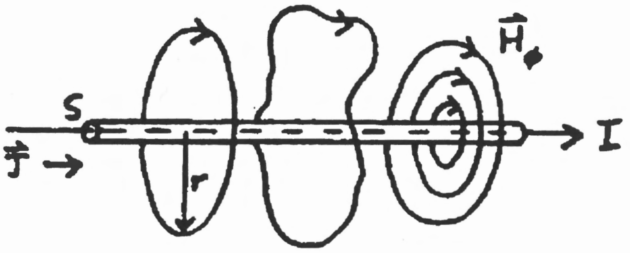
9.1 The line integral of magnetic field

- As a consequence of the work done by Ampère, it can be shown that the line integral of static magnetic field, taken about any closed path, must equal the current that the path encircles. Thus:

$$\oint \mathbf{H} \cdot d\mathbf{l} = I = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{9.1}$$

- Notice that I is positive if the line integral is performed *clockwise*, looking in the direction of I. Notice also the similarity between this law and Gauss's Law in electrostatics: in highly-symmetrical geometries this law, know as *Ampère's circuital law*, can be used to obtain the magnetic field very easily.

- The proof of this law is not given here, as it must wait for a formal understanding of magnetic vector potential.



e.g.1 The magnetic field due to a line current

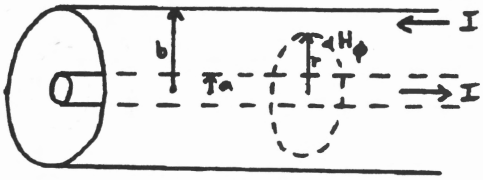
- Already calculated in the previous lecture, we now find this magnetic field by integrating around a circular path of radius r centred on the axis of the wire, as drawn above. Cylindrical co-ordinates is the natural system to use. The symmetry dictates that $\mathbf{H}$ will not vary with the angle ϕ as we integrate around the path, and that $\mathbf{H} \parallel d\mathbf{l}$ everywhere along the circular path. Hence, the integral becomes simply

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_\phi \int d\mathbf{l} = 2\pi r H_\phi = I \tag{9.2}$$

from which $H_\phi = \dfrac{I}{2\pi r}$ and $B_\phi = \dfrac{\mu I}{2\pi r}$, just as we had before.

e.g.2 Magnetic field between coaxial cylinders

- Consider a coaxial cable in which current I travels in the inner cylinder and returns in the outer. Exactly the same development as above, integrating along a circular path that lies between the cylinders, will give



$$H_\phi = \frac{I}{2\pi r} \quad \text{for} \quad a < r < b \tag{9.3}$$

- Note that, if $r > b$, then the total enclosed current is $I - I = 0$, so that $\oint \mathbf{H} \cdot d\mathbf{l} = 0$, from which

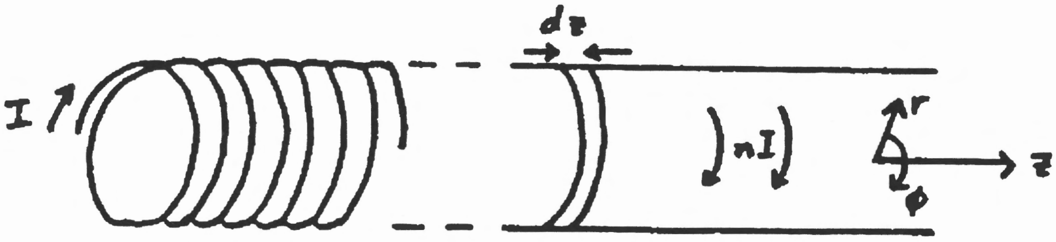
$$\mathbf{H} = \mathbf{B} = 0 \quad \text{for} \quad r > b \tag{9.4}$$

This explains why coaxial cable is used to transmit any signal from which components or systems that lie near the outer sheath of the transmission line need to be electromagnetically shielded.

Eg Money: E.M.C – electro magnetic [illegible]

e.g.3 The magnetic field of a solenoid

- If current I flows in an infinitely-long coil (known as a *solenoid*), which is wound with n turns per metre, then it is acceptable to regard the current as flowing in a *sheet*, entirely circumferentially around the solenoid. We ignore the tiny component of current flow in the z-direction that is due to the progress of the solenoid winding. The sheet's current density is nI Am^{-1}, and so the current that flows in a differential strip of length dz metres is $nI\,dz$ amperes.



- By the methods of the last lecture, we can write the field due to this differential element (which is just a ring of current) as

see proof p57

$$dH_z = \frac{nIa^2\,dz}{2(a^2+z^2)^{\frac{3}{2}}} \tag{9.5}$$

and so, if the solenoid has length $2b$ and is centred at the origin,

$$\begin{aligned} H_z &= \int_{-b}^{b} \frac{nIa^2\,dz}{2(a^2+z^2)^{\frac{3}{2}}} && (9.6)\\ &= \frac{nIa^2}{2}\int_{-b}^{b}\frac{1}{(a^2+z^2)^{\frac{3}{2}}}\,dz && (9.7)\\ &= \frac{nIa^2}{2a^2}\left[\frac{2}{[(a/b)^2+1]^{\frac{1}{2}}}\right] && (9.8)\end{aligned}$$

$= \dfrac{nI}{\sqrt{\frac{a^2}{b^2}+1}}$

let $b \to \infty$

$H \to nI$

$z = a\tan\theta$

$\sqrt{a^2+z^2}$, θ, z, a

$dz = a\sec^2\theta \cdot d\theta$

$\int_{\tan^{-1}\frac{-b}{a}}^{\tan^{-1}\frac{b}{a}}$

$\dfrac{a\sec^2 d\theta}{a^3(1+\tan^2\theta)^{3/2}}$

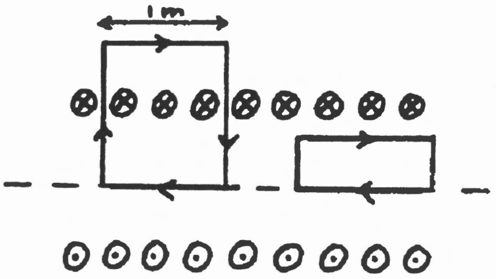
$= \sec^2\theta$

$= a^2\sec^3\theta$

=

- The integral in the line above is relatively easy to obtain, using the substitution $z \equiv a \tan\theta$. Having found the expression for H_z for a finite solenoid, we now let $|b| \rightarrow \infty$ to arrive at a simple expression (purely in the z-direction) for the for the magnetic field of an infinite solenoid at any point on its axis:

$$H_z = nI \qquad \text{on the axis} \tag{9.9}$$



- The diagram shows a sectional view of a part of an infinite solenoid, with the current directions indicated by dots and crosses. Now if we consider the 1-metre-long path shown on the left-hand-side of this diagram, we can use Ampère's circuital law to give us

$$\oint \mathbf{H} \cdot d\mathbf{l} = nI \tag{9.10}$$

  But we already know from the above that $H_z = nI$ on the solenoid's axis, and that since $\mathbf{H}$ is purely $\hat{\mathbf{z}}$-directed there are no contributions to the integral from the radial sections of the path. Thus, all of the contribution to $\oint \mathbf{H} \cdot d\mathbf{l}$ comes from the axial section of the chosen path, and hence *the field outside the solenoid is zero.*

- Furthermore, if we took the other 1-metre-long path shown in the diagram, then by Ampère's circuital law we know that $\oint \mathbf{H} \cdot d\mathbf{l} = 0$ (since the path encloses no current). But we know that $\int \mathbf{H} \cdot d\mathbf{l} = nI$ for the axial section, so the contribution from the other horizontal section must be $-nI$. In addition, we note that the choice of where the upper horizontal section is drawn is arbitrary. In other words, *the magnetic field is constant everywhere inside the solenoid*, and of value nI Am^{-1}.

Chapter 10

Inductance

10.1 Definition of inductance

- In circuit analysis, we would describe any structure which stores energy in a magnetic field as an *inductor*. Recall from basic physics how any current has a *magnetic flux* (Φ) associated with it. Where two (or more) conductors are separated by free space, or by a material medium, the magnetic fluxes due to each conductor are said to be *linked*. Thus, for a coil of N turns for example, the *total flux linkage* is written

$$\lambda = N\Phi \tag{10.1}$$

- Since the flux, Φ, is usually proportional to the current, I, flowing in the inductor, so too is the total flux linkage, λ. Thus we may write

$$\lambda = LI \tag{10.2}$$

where the constant of proportionality, L in *henries*, is known as the *inductance* of whatever arrangement of conductors is being studied.

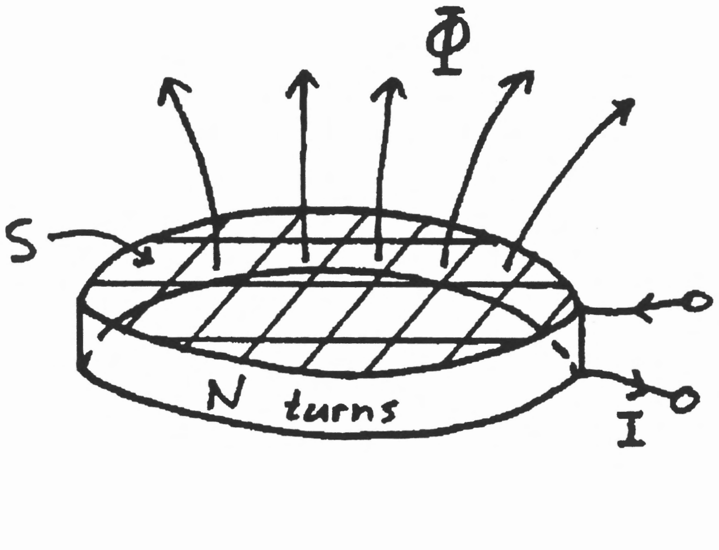
- Now the total flux linkage, λ, can be found by simply adding up the magnetic flux density over the area in which the flux linkage exists:

$$\lambda = \int_S \mathbf{B} \cdot d\mathbf{S} \tag{10.3}$$

So, where the surface S has been specified, we have the important expression for inductance:

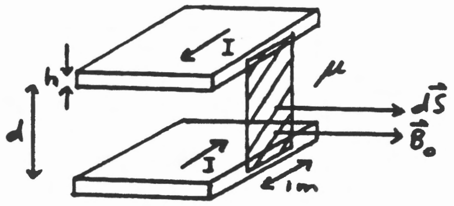
$$L = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{10.4}$$

- Take, for example, the coil of N turns shown here. Almost all of the flux due to the current passes through the shaded surface S. A little, however, passes through the wires of the coil itself. We often distinguish these as giving rise, respectively, to the *external inductance* and to the *internal inductance* of the coil. As suggested in the case of the coil of N turns, the internal inductance may be very much smaller than the external inductance, and is sometimes negligible.

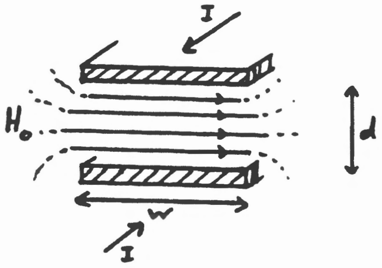


e.g.1 The inductance of a parallel-plate transmission line

- Consider a pair of parallel plates, such as, for instance, in a microstrip transmission line. Let the plates have width w and have separation d. The separating medium has permeability μ. Each strip is of thickness h, considered negligible, and a current I flows in the one strip and returns in the other.



- We make the important assumption that the strips are wide enough, compared to their distance apart, that the magnetic field lines between them are parallel. We write the magnetic field strength between the strips as H_o, and note that we have made the assumption that $d << w$. We also assume that the magnetic field is virtually zero elsewhere - that is, we neglect any fringing field.



- Then, by *Ampère's circuital law*,

$$I = \oint \mathbf{H} \cdot d\mathbf{l} = H_o w \tag{10.5}$$

over a path of total length w between the conductors, and hence

$$B_o = \frac{\mu I}{w} \tag{10.6}$$

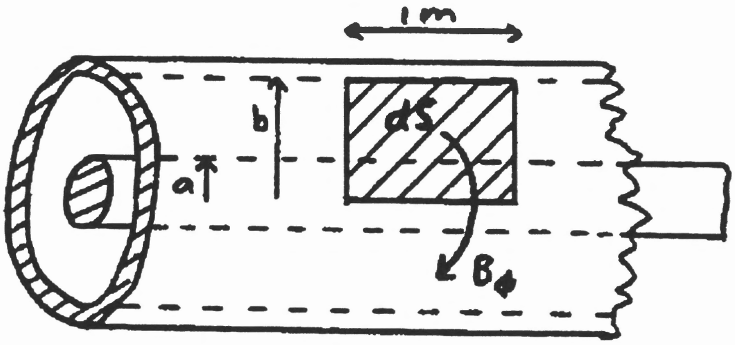
- Now, noting that $\mathbf{B}_o$ is parallel to $d\mathbf{S}$ everywhere over the the one-metre-long flux-gauging surface shown, we can find the inductance per metre of transmission line:

$$L = \frac{1}{I}\int_S \mathbf{B}_o \cdot d\mathbf{S} = \frac{B_o}{I}\int_S d\mathbf{S} = \frac{\mu\, I}{wI}(d \times 1) = \frac{\mu\, d}{w} \ \text{ H/m} \tag{10.7}$$

e.g.2 Inductance of a coaxial transmission line

- We know from previous examples that the magnetic field in a coaxial cable is entirely circumferential, and of value
$$H_\phi = \frac{I}{2\pi r} \quad \text{so} \quad B_\phi = \frac{\mu I}{2\pi r} \tag{10.8}$$
where r lies between the radii of the two conductors in which the current I flows, and μ is the permeability of the medium between the conductors.

- Referring to this diagram of a coaxial cable, the total flux linkage per metre is to be found by integrating the flux density over the area S and noting that $\mathbf{B}$ is parallel to $d\mathbf{S}$ everywhere on the surface.



- Thus,
$$\int_S \mathbf{B} \cdot d\mathbf{S} = \frac{\mu I}{2\pi} \int_a^b \frac{1}{r}\, dr = \frac{\mu I}{2\pi} \ln\left(\frac{b}{a}\right) \tag{10.9}$$
and hence
$$L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{H/m} \tag{10.10}$$

- Note that this expression gives the *external* inductance of the coaxial cable, because it only considers the flux in the region between the two conductors. We make the further assumption (particularly valid at high frequencies, as we shall see) that the internal inductance is negligible.

Chapter 11

The Curl, and Stokes' Theorem

11.1 Using and interpreting curl

- Ampère's circuital law gives rise to useful expressions, in integral form, for the magnetic field vector. In electrostatics, we saw the value of also having compact expressions for unknown quantities in *differential* form, and we introduced the vector operation of *divergence* to achieve this. In a somewhat similar way, the *curl* is used to obtain differential expressions for the magnetic field vectors, **H** or **B**.

- It is assumed that you have worked with curl **F** (written $\nabla \times \mathbf{F}$) before, particularly in MAM280W. If this is not the case, then you are advised to consult mathematics texts to gain a better understanding of the curl operation. Note, in particular, that $\nabla \times \mathbf{F}$ is a *vector*.

- The basic definition of curl **F**, at a point and in the $\hat{\mathbf{x}}$-direction, is *the limit of the line integral of the field* **F** *taken around the point, as the area enclosed by the path, and perpendicular to* $\hat{\mathbf{x}}$, *vanishes to zero.* (Compare our earlier definition of divergence with this). Thus

$$[\text{curl}\,\mathbf{F}]_x \equiv \lim_{\Delta S_x \to 0} \frac{\oint \mathbf{F} \cdot d\mathbf{l}}{\Delta S_x} \tag{11.1}$$

  and this can also be stated in the $\hat{\mathbf{y}}$- and $\hat{\mathbf{z}}$- directions, to give

$$\text{curl}\,\mathbf{F} \equiv \nabla \times \mathbf{F} = [\text{curl}\,F]_x + [\text{curl}\,F]_y + [\text{curl}\,F]_z \tag{11.2}$$

Equivalently, although we will not prove this here,

$$\operatorname{curl}\mathbf{F} \equiv \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{11.3}$$

- WARNING! Unlike divergence, the curl of a vector field does not have an easily-visualised interpretation. Curved field lines may possess zero curl, while some fields of straight, parallel vectors may have non-zero curl. Perhaps the best way to perceive the curl of a vector field at a point is to imagine that all of the field lines are velocity vectors and that a small, circular paddle sits at the point of interest. If the paddle spins, then the curl at that point is non-zero.

- As you might expect, there are equivalent expressions for $\nabla \times \mathbf{F}$ in both cylindrical and spherical co-ordinates, and these are given in formula sheets and (for $\nabla \times \mathbf{H}$) below:

$$\begin{aligned} \nabla\times\mathbf{H} &= \frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi\sin\theta)-\frac{\partial H_\theta}{\partial\phi}\overset{\hat{x}}{\right]}+\frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi}-\frac{\partial}{\partial r}(rH_\phi)\overset{\hat{y}}{\right]}+\frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rH_\theta)-\frac{\partial H_r}{\partial\theta}\overset{\hat{z}}{\right]} \text{ (sph)} \\ \nabla\times\mathbf{H} &= \hat{\mathbf{r}}\left[\frac{1}{r}\frac{\partial H_z}{\partial\phi}-\frac{\partial H_\phi}{\partial z}\right]+\hat{\phi}\left[\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}\right]+\hat{\mathbf{z}}\left[\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r}-\frac{1}{r}\frac{\partial H_r}{\partial\phi}\right] \text{ (cyl)} \end{aligned} \tag{11.4}$$

11.2 The curl of electrostatic field

- Any vector, $\mathbf{F}$, which can be written as the gradient of a scalar function, $\xi(x, y, z)$, has zero curl. This is relatively easy to show, since, if

$$\mathbf{F} = \nabla\xi = \hat{\mathbf{x}}\frac{\partial\xi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\xi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\xi}{\partial z} \tag{11.5}$$

then, by our determinant definition of curl,

$$\nabla \times \mathbf{F} = \hat{\mathbf{x}}\left(\frac{\partial^2\xi}{\partial y\partial z} - \frac{\partial^2\xi}{\partial z\partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2\xi}{\partial z\partial x} - \frac{\partial^2\xi}{\partial x\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial^2\xi}{\partial x\partial y} - \frac{\partial^2\xi}{\partial y\partial x}\right) \tag{11.6}$$

But since, for a general scalar function $f(a, b)$, $\frac{\partial^2 f}{\partial a\partial b} = \frac{\partial^2 f}{\partial b\partial a}$, the equation above becomes simply

$$\nabla \times \mathbf{F} = 0 \tag{11.7}$$

- Now, electrostatic field is the (negative) gradient of a scalar function, known to us as the potential function, $\Phi(x, y, z)$. Thus, knowing that $\mathbf{E} = -\nabla\Phi$, we have
$$\nabla \times \mathbf{E} = 0 \tag{11.8}$$

- You may recall from mathematics that zero curl is a characteristic of a *conservative* field. It is now easy to see why. Since
$$\nabla \times \mathbf{E} \equiv \lim_{\Delta S \to 0} \frac{\oint \mathbf{E} \cdot d\mathbf{l}}{\Delta S} = 0 \tag{11.9}$$
it follows that $\oint \mathbf{E} \cdot d\mathbf{l} = 0$, which is the condition for a conservative field.

11.3 The curl of magnetostatic field

- From the basic definition of curl, we see that the $\hat{\mathbf{x}}$-component of the curl of magnetic field intensity is
$$[\mathrm{curl}\,\mathbf{H}]_x \equiv \lim_{\Delta S_x \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{l}}{\Delta S_x} \tag{11.10}$$

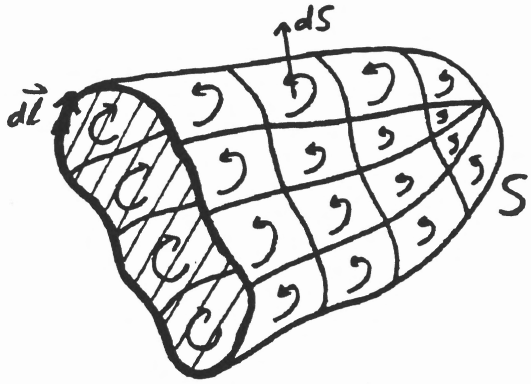
- But we saw in stating Ampère's circuital law how $\oint \mathbf{H} \cdot d\mathbf{l}$ is related to the current density vector $\mathbf{J} = (J_x, J_y, J_z)$ by
$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} = I \tag{11.11}$$
Thus,
$$[\mathrm{curl}\,\mathbf{H}]_x = \lim_{\Delta S_x \to 0} \frac{\int_{S_x} \mathbf{J} \cdot d\mathbf{S}}{\Delta S_x} = J_x \tag{11.12}$$
by definition, and likewise $[\mathrm{curl}\,\mathbf{H}]_y = J_y$ and $[\mathrm{curl}\,\mathbf{H}]_z = J_z$

- Therefore, by equation 11.2, we arrive at the compact and useful conclusion that, for magnetostatic fields,
$$\nabla \times \mathbf{H} = \mathbf{J} \quad \text{or} \quad \nabla \times \mathbf{B} = \mu\mathbf{J} \tag{11.13}$$

11.4 Stokes' Theorem

- Conversion betwen the integral and differential forms of relationships for electric fields was made easier by the divergence theorem. In a similar way, Stokes' Theorem gives us a convenient "short cut" in our vector algebra when handling magnetic fields.

- Imagine a surface, S, embedded in a field, $\mathbf{F}$. We wish to divide the surface into infinitesimal vector areas, $d\mathbf{S}$, and to calculate $(\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ in each area.



- Since, by the definition of curl

$$(\nabla \times \mathbf{F}) \cdot d\mathbf{S} \rightarrow \oint_{d\mathbf{S}} \mathbf{F} \cdot d\mathbf{l} \quad \text{as} \quad d\mathbf{S} \rightarrow 0 \tag{11.14}$$

we see that $(\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ can be found in each little area simply by integrating $\mathbf{F}$ around its small perimeter.

- If we now add up all the contributions of $(\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ for all of the small areas, then we find that at every boundary between small areas the line integrals *cancel*, leaving only the integral of $\mathbf{F}$ along the outer boundary of the surface. Hence

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint \mathbf{F} \cdot d\mathbf{l} \tag{11.15}$$

- This is *Stokes' Theorem.* We can now apply it to magnetic fields and write

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} \tag{11.16}$$

and, since $\nabla \times \mathbf{H} = \mathbf{J}$, we arrive back at a statement of Ampère's circuital law:

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} = I \tag{11.17}$$

e.g. Find the curl of the magnetic field both outside and inside a long, straight wire of radius a, that carries a current I

- Outside the wire, the situation is very straightforward. The magnetic field is circumferential only and we know that $H_\phi = \frac{I}{2\pi r}$, or equivalently, that $\mathbf{H} = \frac{I}{2\pi r}\hat{\boldsymbol{\phi}}$. We can find $\nabla \times \mathbf{H}$ in cylindrical co-ordinates by observing that $\mathbf{H}$ has no $\hat{\mathbf{z}}$- or $\hat{\mathbf{r}}$- component and no $\hat{\mathbf{z}}$- or $\hat{\boldsymbol{\phi}}$-dependency. So

$$\frac{\partial H_z}{\partial \phi} = \frac{\partial H_\phi}{\partial z} = \frac{\partial H_r}{\partial z} = \frac{\partial H_z}{\partial r} = \frac{\partial H_r}{\partial \phi} = 0 \tag{11.18}$$

Also, since $rH_\phi = \frac{I}{2\pi}$, we obtain

$$\frac{\partial(rH_\phi)}{\partial r} = 0 \tag{11.19}$$

Hence, outside the wire all components of the curl are zero, so

$$\nabla \times \mathbf{H} = 0 \quad \text{(outside)} \tag{11.20}$$

- Inside the wire, there is a uniform current density $\mathbf{J} = \frac{I}{\pi a^2}$ acting in the $\hat{\mathbf{z}}$-direction. Considering a circular loop of radius $r < a$ within the wire, together with Ampère's circuital law, reveals that

$$\oint \mathbf{H} \cdot d\mathbf{l} = 2\pi r H_\phi = I_{enc} = \pi r^2 |\mathbf{J}| = \frac{r^2}{a^2} I \tag{11.21}$$

Hence $H_\phi = \frac{Ir}{2\pi a^2}$, which has $\hat{\boldsymbol{\phi}}$-component only, and no $\hat{\mathbf{z}}$-dependency. Thus, in cylindrical co-ordinates,

$$\nabla \times \mathbf{H} = \hat{\mathbf{z}}\left[\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial \phi}\right] = \hat{\mathbf{z}}\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{Ir^2}{2\pi a^2}\right] = \left(\frac{I}{\pi a^2}\right)\hat{\mathbf{z}} \tag{11.22}$$

and so, as we should expect from our earlier equations,

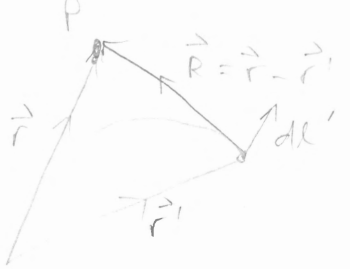
$$\nabla \times \mathbf{H} = \left(\frac{I}{\pi a^2}\right)\hat{\mathbf{z}} = \mathbf{J} \quad \text{(inside)} \tag{11.23}$$

Write out and try this

Six terms ~~are~~ all [illegible] dependency

H only has ϕ component [illegible]

ϕ component [illegible]



Chapter 12

Magnetic Potential

12.1 Defining vector magnetic potential

- Just as we were able to calculate electrostatic fields knowing the electric potential function Φ and using $\mathbf{E} = -\nabla\Phi$, so it is useful to define a *magnetic potential*, $\mathbf{A}$, from which magnetic flux density can be calculated. Magnetic potential is a *vector*, which arises because of a current flowing: each element of the vector $\mathbf{A}$ has the direction of the small element of current path that produces it, and we can evaluate $\mathbf{A}$ at a point by integrating the contribution from all the elements of the path.

- We start with Ampère's law

$$\mathbf{H}(\mathbf{r}) = \int \frac{I'(\mathbf{r}')\, d\mathbf{l}' \times \mathbf{R}}{4\pi R^3} \quad \text{or} \quad \mathbf{B}(\mathbf{r}) = \mu \int \frac{I'(\mathbf{r}')\, d\mathbf{l}' \times \mathbf{R}}{4\pi R^3} \tag{12.1}$$

which expresses the field intensity at a point r due to a current I flowing in a loop l, with each element of the loop at $\mathbf{r}'$ denoted $d\mathbf{l}'$. Recall that the vector $\mathbf{R}$ is just

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \hat{\mathbf{x}}(x - x') + \hat{\mathbf{y}}(y - y') + \hat{\mathbf{z}}(z - z') \tag{12.2}$$

Then you can use the definition of the gradient operator to show that, in rectangular co-ordinates (try this on paper),

$$\frac{d\mathbf{l}' \times \mathbf{R}}{R^3} = \nabla\left(\frac{1}{R}\right) \times d\mathbf{l}' \tag{12.3}$$

learn this on

- Now, for any vector $\mathbf{F}$ and scalar function ψ in rectangular co-ordinates, you can also show (and should do this for yourself) that

$$\nabla\times(\psi\mathbf{F}) = \psi\nabla\times\mathbf{F} - \mathbf{F}\times\nabla\psi \tag{12.4}$$

Hence, replacing ψ with $\frac{1}{R}$ and $\mathbf{F}$ with $d\mathbf{l}'$, we have that

$$\nabla\times\frac{d\mathbf{l}'}{R} = \frac{1}{R}\nabla\times d\mathbf{l}' - d\mathbf{l}'\times\nabla\left(\frac{1}{R}\right) \tag{12.5}$$

so, recalling that for vectors $\mathbf{V}_1$ and $\mathbf{V}_2$, $\mathbf{V}_1\times\mathbf{V}_2 = -\mathbf{V}_2\times\mathbf{V}_1$, we get

$$\nabla\left(\frac{1}{R}\right)\times d\mathbf{l}' = \nabla\times\left(\frac{d\mathbf{l}'}{R}\right) - \frac{1}{R}\nabla\times d\mathbf{l}' \tag{12.6}$$

- Back in Ampère's law, this means that we can replace $\frac{d\mathbf{l}'\times\mathbf{R}}{R^3}$ with

$$\nabla\left(\frac{1}{R}\right)\times d\mathbf{l}' = \nabla\times\left(\frac{d\mathbf{l}'}{R}\right) - \frac{1}{R}\nabla\times d\mathbf{l}' = \nabla\times\left(\frac{d\mathbf{l}'}{R}\right) \tag{12.7}$$

since $\nabla\times d\mathbf{l}' = 0$ (because $d\mathbf{l}'$ is not a function of x, y or z and so has partial derivatives of zero with respect to each of these variables, so that its curl is zero too in rectangular co-ordinates).

- Hence, Ampère's law becomes:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi}\int I'(\mathbf{r}')\nabla\times\left(\frac{d\mathbf{l}'}{R}\right) = \nabla\times\mathbf{A}(r) \tag{12.8}$$

if we define

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int\frac{I'(\mathbf{r}')\,d\mathbf{l}'}{R} \tag{12.9}$$

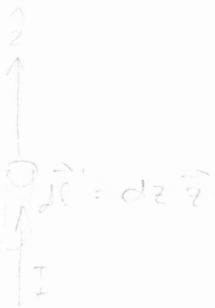
We call $\mathbf{A}$ the *vector potential* of the magnetic field. In the case where we only know the current density $\mathbf{J}$ through the source of magnetic field, since $I = J\,dS$ where $d\mathbf{S}\perp\mathbf{J}$, and since $d\mathbf{l}'\parallel\mathbf{J}$, dS and dl' form a volume element dV', we can thus also write

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{r}')\,dV'}{R} \tag{12.10}$$

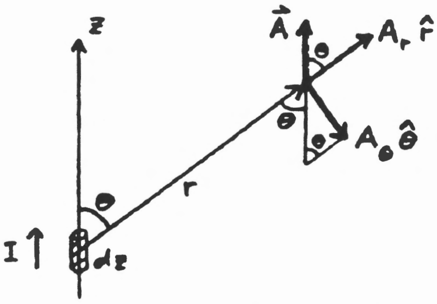
12.2 Using B = ∇ × A

- Visualising magnetic vector potential is extremely difficult to do. Instead, we just work with **A** as a useful intermediate function and remember that $\mathbf{B} = \nabla \times \mathbf{A}$. This may make our work in calculating **B** actually easier.

- Thus, for example, to calculate the magnetic field due to a single element of current I, flowing in the z-direction, we could first obtain



$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu I'(\mathbf{r}')\,d\mathbf{l}'}{4\pi R} = \int \frac{\mu I\,dz}{4\pi r}\hat{\mathbf{z}} = \hat{\mathbf{z}}A_z \qquad (12.11)$$



We then observe that **A** is in the z-direction but is dependent only on radial distance r. To obtain $\nabla \times \mathbf{A}$, we will need to get **A** expressed in a suitable co-ordinate system, which would be spherical co-ordinates in which $A_r = A_z \cos\theta$ and $A_\theta = -A_z \sin\theta$. Note that θ is the angle that the vector from the source to the point of interest makes with the direction of the current in the source. Note also that $A_\phi = 0$ since the current element has circular symmetry in the ϕ direction.

- Using the definition of $\nabla \times \mathbf{A}$ in spherical co-ordinates, we therefore have

watch out error here ✓

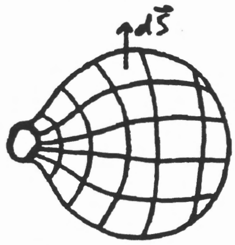
$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right] = \hat{\phi}\int\left(\frac{\mu I\,dz\,\sin\theta}{4\pi r^2}\right) \qquad (12.12)$$

But this is precisely $\frac{\mu}{4\pi R^3} I\,|dz|\,|R|\,\sin\theta$, acting perpendicularly to both $\hat{\mathbf{z}}$ and $\hat{\mathbf{r}}$, as you would obtain from applying Ampère's law directly. This illustrates that $\mathbf{B} = \nabla \times \mathbf{A}$ can be used as a possible alternative route for calculating magnetic field.

learn this

12.3 The divergence of magnetic flux density

- We have already shown that, for a vector function **F** and scalar function ξ, where $\mathbf{F} = \nabla\xi$, $\nabla \times \mathbf{F} = 0$.



- Consider now Stokes's theorem for a surface S bounded by a short closed line, as shown. Suppose the surface to be immersed in a field **F**. Then

$$\oint \mathbf{F} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \qquad (12.13)$$

As we let the boundary line shrink to zero, the surface S becomes a closed surface, and the line integral shrinks to zero. Hence

$$\oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0 \qquad (12.14)$$

regardless of the nature of **F** or of the closed surface S. We can now apply the *divergence theorem* for any vector **G**

$$\oint_S \mathbf{G} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{G}\, dV \qquad (12.15)$$

to obtain, with $\mathbf{G} = \nabla \times \mathbf{F}$,

$$\oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_V \nabla \cdot \nabla \times \mathbf{F}\, dV = 0 \qquad (12.16)$$

Hence, in this completely general case we have established that

$$\nabla \cdot \nabla \times \mathbf{F} = 0 \qquad (12.17)$$

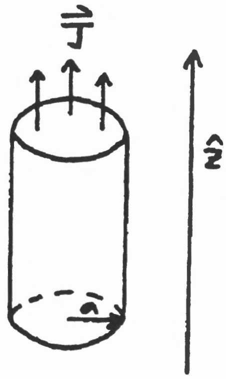
for any vector field **F**.

- Now, since $\mathbf{B} = \nabla \times \mathbf{A}$, we can therefore write

$$\nabla \cdot \nabla \times \mathbf{A} = \nabla \cdot \mathbf{B} = 0 \qquad (12.18)$$

This extremely important relationship tells us that there are no sources of magnetic flux, and therefore nothing which corresponds in magnetostatics to the phenomenon of charge in electrostatics. Put another way, $\nabla \cdot \mathbf{B} = 0$ is a statement that magnetic field lines will always form closed loops.

e.g. From the vector potential, find the magnetic field due to a uniform, circularly-symmetric axial current density



- We are considering, in effect, the field inside a wire due to a current I flowing in the wire. Let the wire have radius a, and let the current density be $\mathbf{J}_o = \frac{I}{\pi a^2}$, acting in the $\hat{\mathbf{z}}$-direction. We have already calculated $\mathbf{H}$ in such a wire, using Ampère's circuital law, but here we will obtain the same result by means of calculating the vector potential $\mathbf{A}$.
- Now, it can be shown that the magnetic vector potential inside the wire obeys Poisson's equation, $\nabla^2\mathbf{A} = -\mu\mathbf{J}$, and that in this case $\mathbf{A}$ is purely $\hat{\mathbf{z}}$-directed and of value

  $$A_z = \frac{-\mu J_o}{4}\left(x^2+y^2\right) \tag{12.19}$$

  (You can check that $\nabla^2\mathbf{A} = -\mu\mathbf{J}$ in this case.) Then

  $$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu}(\nabla\times\mathbf{A}) \tag{12.20}$$

  $$= \frac{1}{\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{-\mu J_o}{4}(x^2+y^2) \end{vmatrix} \tag{12.21}$$

  $$= \frac{1}{\mu}\left[\hat{\mathbf{x}}\left(\frac{-\mu J_o}{2}y\right) - \hat{\mathbf{y}}\left(\frac{-\mu J_o}{2}x\right)\right] \tag{12.22}$$

  $$= \frac{-J_o}{2}\left(\hat{\mathbf{x}}y - \hat{\mathbf{y}}x\right) \tag{12.23}$$

- Converting this result to cylindrical co-ordinates, in which $x = r\cos\phi$, $y = r\sin\phi$ and $\hat{\boldsymbol{\phi}} = \frac{1}{r}\left(\frac{\partial x}{\partial\phi}\hat{\mathbf{x}} + \frac{\partial y}{\partial\phi}\hat{\mathbf{y}}\right) = \sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$ we obtain

$$x\hat{\mathbf{y}} - y\hat{\mathbf{x}} = r\hat{\boldsymbol{\phi}} \tag{12.24}$$

and so

$$\mathbf{H} = \frac{J_0 r}{2}\hat{\phi} = \frac{Ir}{2\pi a^2}\hat{\phi} \tag{12.25}$$

exactly as we had before.

12.4 Scalar magnetic potential

- It is sometimes convenient to work with another quantity, known as the *scalar magnetic potential*, and written Φ_m, by analogy with scalar electric potential Φ_e. In *current-free* regions we can write (as we have seen)

$$\nabla \times \mathbf{H} = 0 \tag{12.26}$$

- We have also seen that any vector that can be expressed as the gradient of a scalar function has zero curl. The reverse is also true, and so we can write $\mathbf{H}$ as the gradient of a scalar function, and this is Φ_m. Thus

$$\mathbf{H} = (-)\nabla\Phi_m \tag{12.27}$$

- Then, since $\nabla \cdot \mathbf{B} = 0$, we can write

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mu\mathbf{H} = \nabla \cdot (-\mu\nabla\Phi_m) \tag{12.28}$$

and so

$$\nabla^2\Phi_m = 0 \tag{12.29}$$

from which we can also see that Φ_m, like Φ_e, obeys Laplace's equation. There are methods available (as we saw in electrostatics) for solving Laplce's equation, so yet another way of calculating magnetic field may be to obtain Φ_m in a given situation, and then to find $\mathbf{H} = -\nabla\Phi_m$.

Chapter 13

Energy in Magnetic Fields

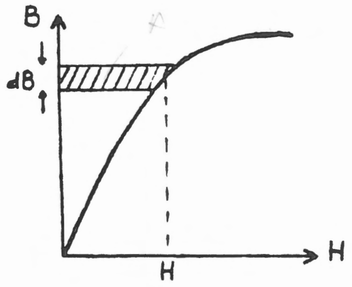
13.1 Energy storage in static magnetic fields

- You may recall that we derived this expression for the energy contained in a small volume of a region in which there is an electrostatic field: $dU_E = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dV$. One can reason in a directly analogous way with static magnetic fields, provided that $\mathbf{B}$ and $\mathbf{H}$ are proportionately related—that is, provided that μ is *constant* for the material medium, in which case the medium is described as being *linear*. We may then write $\mathbf{B} = \mu\mathbf{H}$ where μ is a constant and, with this proviso, we have (after integration and if $\mathbf{B} \parallel \mathbf{H}$):

$B \parallel H$
– isotropic

$$U_H = \frac{1}{2}\int_V \mathbf{B}\cdot\mathbf{H}\,dV = \frac{\mu}{2}\int_V H^2\,dV \tag{13.1}$$

- Many magnetic materials are not linear—in other words, the value of μ depends upon the strength of the magnetic field applied. The result is that, if you plot $|\mathbf{B}|$ against $|\mathbf{H}|$ as the field strength varies in this material, you will obtain a curve like the one shown. As we shall see, this complicates the picture of the energy stored in the field. In the non-linear case, the extra energy stored in a volume V for a small increase in $\mathbf{B}$ is



$$dU_H = \int_V \mathbf{H}\cdot d\mathbf{B}\,dV \tag{13.2}$$

Proof

TEST QUESTION

If $U_H = \frac{1}{2}\int_V \vec{B}\vec{H}\,.dV$

then $dU_H = \frac{1}{2}\int_V (\vec{B}.d\vec{H} + \vec{H}.d\vec{B})\,dV = \frac{1}{2}\int_V (\mu\vec{H}.\frac{d\vec{B}}{\mu} + \vec{H}.d\vec{B})\,dV$

$= \int_V \vec{H}.d\vec{B}\,.dV$

- If **H** || **B**, which is a feature of *isotropic* materials, we can then write

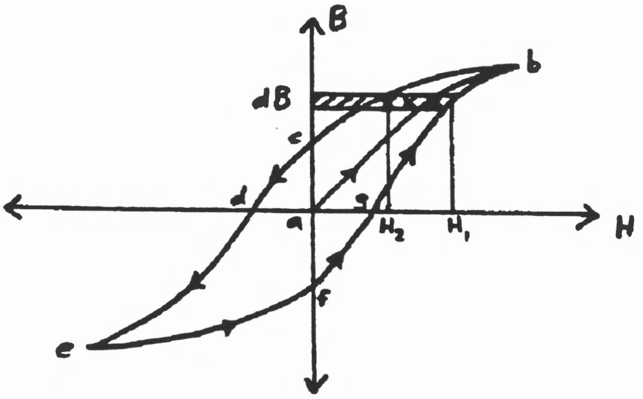
$$dU_H = H\,dB \int_V dV \tag{13.3}$$

$= A \int_V dV$

which means that the shaded area in the graph above represents the extra energy stored per unit volume for a small increment in the **B**-field.

13.2 Energy loss in hysteretic materials

- We have therefore seen that a non-linear, isotropic material will respond to an increase in field strength from zero by travelling from a to b in the B-H graph shown below.



- By the time the material is brought to point b, small magnetic *domains* within the material may have been aligned by the applied field. There is a tendency for this alignment to be semi-permanent, with the result that the material is somewhat *magnetised*, i.e. it will act as a permanent magnet.

- If the applied field is now decreased, the material does *not* retrace the path from b to a, but will travel the path cde as the applied field is brought first to zero and is then made negative. Note that at point c, where the applied **H**-field is zero, there nevertheless remains a **B**-field due to the magnetisation of the material. This is known as the *remanence* or *retentivity* of the material. It then requires a further applied field in the negative direction to bring the **B**-field to zero at point d. This value of **H** is known as the *coercive force*.

- Similar arguments apply by symmetry as the applied **H**-field is again increased, and it is found that the material will trace a path from e through f and g back to b. The tendency of the material to "remember" its most recent state of magnetisation and not to retrace its **B**-**H** path until it has traversed the full loop $bcdefg$ is known as *hysteresis*, and the loop is called a *hysteresis* loop.

- Applying our law for energy storage as the **B**-field varies, we see, for example, that as the material moves from g to b there is a point (shown on the figure) where the increase in **B**-field leads to the storage of energy $H_1\,dB$ per unit volume. Then, as the material moves from b to c, at the same **B**-field strength there is a release of stored energy equal to $H_2\,dB$. The difference between the energy added to the stored energy and the energy taken from storage is $(H_1 - H_2)\,dB$, which is shown in cross-hatched shading on the diagram. This energy is effectively wasted, being dissipated as heat during the hysteresis process.

- It should therefore be clear that in one transit of the hysteresis loop $bcdefgb$, the *energy dissipated per unit volume* of the material is equal to the area enclosed by the hysteresis loop. Materials with narrow hysteresis loops are termed "soft", and are of value in electromagnets, transformers and motors because their energy wastage is relatively small. "Hard" magnetic materials have broad hysteresis loops and are used for making permanent magnets.

13.3 Magnetic field energy and inductance

- In the analysis of linear circuits, we are familiar with the expression $\frac{1}{2}LI^2$ for the energy stored in the magnetic field of a circuit element. Since we have already seen how to calculate the inductance of certain symmetrical structures directly from field considerations, we now have the opportunity to check that our two definitions for stored energy are the same.

- The examples which follow will demonstrate and explore the useful relationship:
$$\int_V \frac{\mu}{2} H^2 \, dV = \frac{1}{2} L I^2 \tag{13.4}$$

- Note carefully that the inductance L refers to the *external* inductance of the structure if we consider only the field $\mathbf{H}$ that lies *outside* the conductors, and refers to the *internal* inductance if we consider only the $\mathbf{H}$-field *within* the conductors. Because the external field generally occupies a much larger volume than the internal field, the external inductance usually far outweighs the internal inductance. The equation above applies equally to both situations, however, and, because of linearity, will also apply for the *total inductance*, if the whole volume both within *and* outside the conductors is considered.

e.g.1 Show that the relation $\int_V \frac{\mu}{2} H^2 \, dV = \frac{1}{2} L I^2$ holds for the region between the two conductors in a coaxial cable

- We already know from Ampère's circuital law that the external inductance is $L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$ H/m and that the field between the conductors is $H_\phi = \frac{I}{2\pi r}$. For a 1-metre-long cylinder $V = \pi r^2 \times 1$, and so $dV = 2\pi r \, dr$.

- Hence, if we consider a 1-metre length of coaxial cable which has an inner conductor of radius a and outer conductor of inner radius b:

$$\begin{aligned}
\int_V \frac{\mu}{2} H^2 \, dV &= \frac{\mu I^2}{2} \int_V \frac{1}{(2\pi r)^2} \, dV = \frac{\mu}{2} I^2 \int_a^b \frac{2\pi r}{(2\pi r)^2} \, dr \\
&= \frac{\mu I^2}{2 \times 2\pi} \int_a^b \frac{1}{r} dr = \frac{I^2}{2} \times \frac{\mu}{2\pi} (\ln|b| - \ln|a|) \\
&= \frac{I^2}{2} \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) = \frac{1}{2} L I^2
\end{aligned} \tag{13.5}$$

Test - do this backwards

e.g.2 Find the inductance per unit length of an infinite solenoid of radius a which has n turns per metre and is filled with a material of permeability μ

- We recall that the field is $H_z = nI$ everywhere inside an infinite solenoid. The volume per unit length is $V = \pi a^2 \times 1$. Hence for any one-metre length we can write the stored energy as

$$\int_V \frac{\mu}{2} H^2 \, dV = \frac{\mu}{2} n^2 I^2 V = \frac{\mu}{2} n^2 I^2 \pi a^2 \qquad (13.6)$$

- But the energy stored is also $\frac{1}{2} L I^2$, so

$$\frac{1}{2} L I^2 = \frac{\mu}{2} n^2 I^2 \pi a^2 \qquad (13.7)$$

  from which

$$L = \mu n^2 \pi a^2 = \mu n^2 A \quad [\mathrm{H/m}] \qquad (13.8)$$

  where A is the cross-sectional area of the solenoid.

- Note that the inductance of a coil is related to the *square* of the number of turns per metre.

Part 3

Maxwell's Equations

Chapter 14

Time-Varying Electric & Magnetic Fields

14.1 Summary of important relationships

- Up to this point, our analysis of fields has included no *dynamic* effects. In all of our examples, the sources of electric or magnetic field have fallen into one of the following categories:

    1. They have been *stationary* in space and *invariant* in time.
    2. They have varied in time very slowly (or at very low frequency), so that over a given period of interest they have been effectively *static*.
    3. They have moved in space, but, as with a steady current in a conductor, this movement has led to no change in the overall field. The field in this situation may be described as *quasistatic*.

- Under static or quasistatic conditions, we have identified or derived a number of important definitions and relationships between field quantities. These are summarised on the next page:

Major Definitions

1. Electric field: $\mathbf{E} = \frac{\mathbf{f}}{q}$ N/C

$d\vec{f} = d\vec{q} \times \vec{F}$

2. Conduction current: $\mathbf{J} = \sigma\mathbf{E}$ A/m^2
   (in a conductor)

3. Convection current: $\mathbf{J} = \rho\mathbf{v}_\rho$ A/m^2
   (in space or a medium)

4. Permittivity: $\epsilon = \epsilon_r\epsilon_o = \frac{\mathbf{D}}{\mathbf{E}}$ F/m
   (We assume ϵ to be a constant, because we assume the material medium to be *homogeneous, isotropic, linear* and *time-invariant.*)

5. Permeability: $\mu = \mu_r\mu_o = \frac{\mathbf{B}}{\mathbf{H}}$ H/m
   (Although this is often not the case in practice, here we also frequently assume that the material medium is *homogeneous, isotropic, linear* and *time-invariant.*)

$\vec{D} = \varepsilon\vec{E}$

Vector relationships for static fields

[illegible]

1. $\nabla \cdot \mathbf{D} = \rho$ (encapsulates Gauss's law and our understanding of charge and flux);

2. $\nabla \cdot \mathbf{B} = 0$ (expresses the absence of magnetic field sources that would be analogous to electric charge);

3. $\nabla \times \mathbf{E} = 0$ (describes the conservative nature of electrostatic field);

4. $\nabla \times \mathbf{H} = \mathbf{J}$ (the differential form of Ampère's circuital law, relating magnetic field and current density).

14.2 Time variation in fields

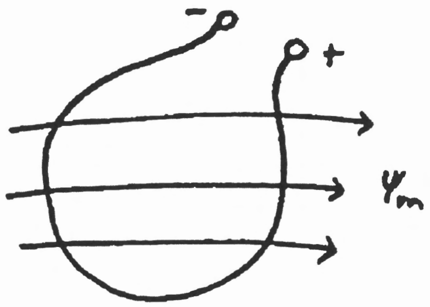
- It is an observed fact, first noted by Faraday, that a time-varying magnetic field will generate an electric field. There is also a complementary effect by which a time-varying electric field gives rise to a magnetic field, which was first quantified by Maxwell.
- It is easy to show that changing magnetic field produces an electric field. You can move a conductor in the vicinity of a magnet and measure the voltage induced in it using an oscilloscope. Alternatively, you can put the conductor near an electromagnet and get the same effect by varying the strength of the electromagnet. It is less easy to show how magnetic field results from a changing electric field, but we will shortly see that this is indeed the case, and we will be able to quantify the effect.
- Our static equations make no mention of the interdependence of changing electric and magnetic fields. We will now develop some ways of expressing this interdependency, with the aim of refining our vector relationships so that they take the effects of time-varying fields into account.

14.3 Faraday's law of electromagnetic induction

- Faraday found that the voltage induced in a conductor that is linked by a time-varying magnetic field is

$$V = \frac{\partial \Psi_m}{\partial t} \tag{14.1}$$

where Ψ_m is the magnetic flux linking the conductor at any instant.



- In the case of a coiled conductor with n turns in the coil, the inductive effect is *additive*, so we write

$$V = n\frac{\partial \Psi_m}{\partial t} \tag{14.2}$$

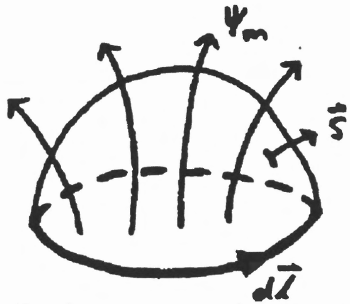
  Note that the magnetic flux that links the coil may change if

  - *either* a nearby source of magnetic flux (e.g. an electromagnet) varies in strength or position,
  - *or* the coiled conductor is moved in the magnetic field.

- Note also that it is not necessary to have a *conductor* for a change in magnetic field to produce a voltage. The same voltage would be induced between points in an insulator, or even in free space, by the same change in magnetic field.

- When there is a contribution to voltages in a region due to changing magnetic flux, it is no longer true that the line integral of electric field is independent of the path (i.e. the electric field is no longer conservative). Instead, we define, for any *closed* path, the *electromotive force* which is the extra induced voltage:

$$\text{emf} = -\oint \mathbf{E}\cdot d\mathbf{l} = \frac{\partial \Psi_m}{\partial t} \tag{14.3}$$



- Then, since the flux linking a loop, Ψ_m, is simply the integral of the normal components of flux density through any surface bounded by the loop, we can write

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial \Psi_m}{\partial t} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{14.4}$$

- Applying Stokes's theorem to the left-hand term above, we get

  $\int_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{l}$

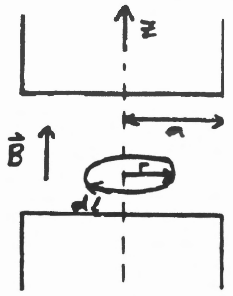
  $$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \qquad (14.5)$$

  and, for this to be true for any surface S, we can set equal the quantities under the integrals, and write

  $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's Law} \qquad (14.6)$$

- This supports our view of *electrostatic* fields as *conservative* because it confirms that $\nabla \times \mathbf{E} = 0$ (if $\frac{\partial \mathbf{B}}{\partial t} = 0$). However, in this more general expression, we see clearly that if nearby magnetic field is time-varying, then the line integral of electric field around a closed path is *non-zero*—this is because some energy is stored in (or delivered by) the magnetic field as it changes in strength. Thus, net work would be done on or by a charge moving around such a closed path.

e.g. Find the greatest rate at which the magnetic field of this large electromagnet can be varied from 0 to 10 T without breakdown of the surrounding air occurring



- The electromagnet has circular poles of radius 20 cm, and we assume that the magnetic field between the poles is uniform. The field elsewhere is taken to be zero. All magnetic flux thus passes at right angles through the circular ends of each pole.

- For any loop of radius r between the poles as shown, Faraday's law

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{14.7}$$

  gives

$$2\pi r E_\phi = -\pi r^2 \frac{\partial B_z}{\partial t} = -\frac{\pi r^2 B_{max}}{\tau} \tag{14.8}$$

  where $B_{max} = 10\,\mathrm{T}$ and τ is the time taken to raise the field to that value from zero. Note that, since $E_\phi \propto r$ in this expression, the greatest E-field will occur at the greatest possible value of r between the poles, i.e. at $r = 20\,\mathrm{cm}$.

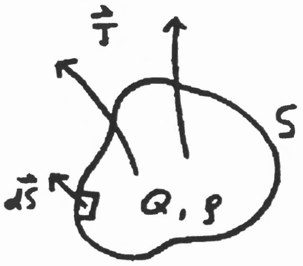
- Since air breaks down at about 3000 kV/m, we can now write

$$\tau = \frac{\pi r^2 B_{max}}{2\pi r |E_\phi|} = \frac{r B_{max}}{2|E_\phi|} = \frac{0.2 \times 10}{2 \times 3 \times 10^6} = \frac{1}{3} \quad \mu\mathrm{s} \tag{14.9}$$

  This is the shortest time in which the magnetic field can be varied from 0 to 10 T without creating an electric arc in the region of the poles at $r = 20\,\mathrm{cm}$.

Chapter 15

Continuity and Displacement Current



15.1 The equation of continuity

- A fundamental principle, that is never seriously challenged, is that of *continuity*. Since current is comprised of moving charge, if a current flows out of a region (through the region's bounding surface), then the positive charge inside the region must decrease. Thus

charge density

$$\oint \mathbf{J} \cdot d\mathbf{S} = I = -\frac{dQ}{dt} = -\frac{\partial}{\partial t}\int \rho\, dV \tag{15.1}$$

principle of continuity integral form

Dividing through by the region's volume, ΔV, and letting $\Delta V \to 0$, we get

$$\lim_{\Delta V \to 0} \frac{\oint \mathbf{J} \cdot d\mathbf{S}}{\Delta V} = \lim_{\Delta V \to 0}\left(-\frac{\partial}{\partial t}\frac{\int \rho\, dV}{\Delta V}\right) \tag{15.2}$$

- This is recognised, through the definition of divergence, to be the same as the *equation of continuity*:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{15.3}$$

differential form

in which ρ is the *net charge density* (not just the density of mobile charge) in the small region of interest.

- Note that *in conductors* $\partial\rho/\partial t$ is non-zero only for very brief transient periods. This can be seen by imagining that a solid conductor has a net charge density of $\rho = \rho_o$ at some time $t = 0$ in some region of itself. Then, since $\mathbf{J} = \sigma\mathbf{E} = \sigma\mathbf{D}/\epsilon$, the equation of continuity gives

  constitutive relationship

  $$\nabla\cdot\mathbf{J} = \nabla\cdot\frac{\sigma}{\epsilon}\mathbf{D} = -\frac{\partial\rho}{\partial t} \qquad (15.4)$$

  Since σ and ϵ are constants, and recalling the relationship $\nabla\cdot\mathbf{D} = \rho$, this leads to

  $$\frac{\sigma}{\epsilon}(\nabla\cdot\mathbf{D}) = \frac{\sigma\rho}{\epsilon} = -\frac{\partial\rho}{\partial t} \qquad (15.5)$$

  and hence

  $$\frac{\partial\rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0 \qquad (15.6)$$

- This is a familiar first order linear differential equation in standard form, for which the solution is

  $$\rho = \rho_o e^{-(\sigma/\epsilon)t} \qquad (15.7)$$

  and so we see that the net charge density, ρ, will decay from its initial value of ρ_o *exponentially*, and with a time constant or *relaxation time* of ϵ/σ seconds. This is the time after which only 36.8% of the initial net charge density remains: in conductors it is very brief. For example, the relaxation time for silver is $\epsilon/\sigma \approx 1.44\times10^{-19}$s. Clearly, after a very short time (e.g. 10^{-17}s), both ρ and $\partial\rho/\partial t$ will be virtually zero.

- Thus, for good conductors only, we may write

  $$\nabla\cdot\mathbf{J} \approx 0 \qquad (15.8)$$

  You may recognise this as the field equivalent of Kirchhoff's Current Law, which states that the net current leaving a node where several perfect conductors meet is equal to zero.

15.2 The definition of displacement current

- We know that, for a static magnetic field **H** in an arbitrary medium, we can write

  $$\mathbf{J}_c = \nabla \times \mathbf{H} \quad \text{Ampere's law} \tag{15.9}$$

  where $\mathbf{J}_c$ is the current density due to the actual motion of charges in the medium. We call $\mathbf{J}_c$ the *conduction current density*. Now, if the magnetic field were time-varying, then there would also be time-varying net charge density in the medium because of the induced currents that would participate in the electromagnetic variations (i.e. $\partial\rho/\partial t \neq 0$)

- However, taking the divergence of equation 15.9, we see that

  $$\nabla \cdot \mathbf{J}_c = \nabla \cdot \nabla \times \mathbf{H} \equiv 0 \neq -\frac{\partial \rho}{\partial t} \tag{15.10}$$

  Hence, the equation $\nabla \times \mathbf{H} = \mathbf{J}_c$ cannot apply to time-varying magnetic fields, because it leads to a contradiction with the equation of continuity.

- Maxwell postulated that time-varying magnetic fields cause the flow of another type of current, called *displacement current*, whose density, the *displacement current density*, he defined as

  $$\mathbf{J}_d \equiv \frac{\partial \mathbf{D}}{\partial t} \tag{15.11}$$

  Maxwell imagined that this extra (unmeasurable) current density forms part of the response to a time-varying magnetic field, and so he amended the purely static relationship $\nabla \times \mathbf{H} = \mathbf{J}_c$ to read

  $$\nabla \times \mathbf{H} = \mathbf{J}_c + \mathbf{J}_d \tag{15.12}$$

  thereby generalising the equation to the dynamic case.

- Now, with
$$\mathbf{J}_c = \nabla \times \mathbf{H} - \mathbf{J}_d \tag{15.13}$$
as Maxwell proposed, we can take the divergence of the equation and obtain
$$\nabla \cdot \mathbf{J}_c = \nabla \cdot (\nabla \times \mathbf{H}) - \nabla \cdot \mathbf{J}_d = 0 - \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = -\frac{\partial \rho}{\partial t} \tag{15.14}$$
and so the new equation does *not* contradict the equation of continuity.

- We can therefore replace the fourth static equation from the last lecture ($\nabla \times \mathbf{H} = \mathbf{J}$) with the improved relationship:
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{15.15}$$
bearing in mind that we revert to the static case whenever the electric field is *not changing* because we then have $\frac{\partial \mathbf{D}}{\partial t} = 0$. Wherever we know that electric flux lines vary in a region over time, however, the displacement current term gains importance, and the new relationship should be used.

15.3 Maxwell's equations

- We are now in a position to re-list the four vector relationships that we established for static fields and stated in the last lecture. The third and fourth equations are modified to take time-variation of fields into account. These four equations, taken together, are always referred to as *Maxwell's Equations*, and they summarise what is known about electromagnetic fields. We will spend the rest of this course using Maxwell's equations to gain a clearer understanding of certain important electromagnetic phenomena. The equations are:

  1. $\nabla \cdot \mathbf{D} = \rho$
  2. $\nabla \cdot \mathbf{B} = 0$
  3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
  4. $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

$\nabla \cdot \vec{E} \cdot \rho$ $\quad \oint \vec{F} \cdot d \quad \nabla \cdot \vec{F} \cdot dV$

$\int_V \nabla \cdot \vec{D} \cdot dV = \int_V \rho \, dV$

$= \oint_S \vec{D} \cdot d\vec{S}$

$\int (\nabla \times E) \cdot ds = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S} = \oint E \cdot dl$

- Note that each of these equations may also be expressed in integral form, either by applying the divergence theorem or else by using Stokes's theorem. You ought to practise performing all of these conversions in either direction. The integral forms of Maxwell's equations are:

  1. $\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV$ $= Q$ (Gauss's Law)
  2. $\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$ (No Magnetic Charges)
  3. $\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$ (Faraday's Law)
  4. $\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{\partial}{\partial t} \int_S \mathbf{D} \cdot d\mathbf{S}$ (Ampère's Law)

$\nabla \times H = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Longleftrightarrow \oint H \cdot dl = \int_S \vec{J} \cdot ds + \frac{\partial}{\partial t} \int_S \vec{D} \cdot ds$

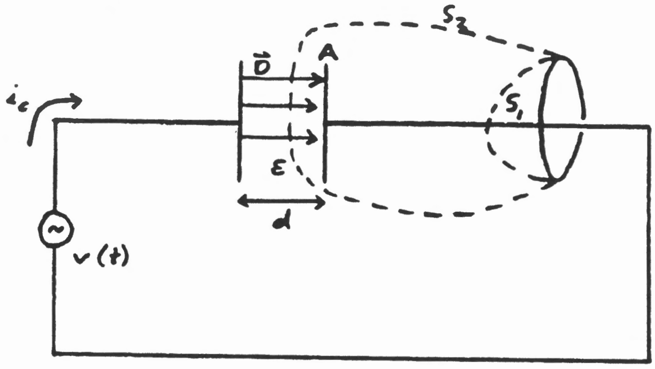
Stokes

Stokes: $\int_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{l}$

Divergence: $\int_V \nabla \cdot \vec{F} \cdot dV = \oint_S F \cdot d\vec{S}$

15.4 Visualising displacement current

- The fourth equation above is a refined version of Ampère's circuital law, applicable to the time-varying case. We will take a further look at it to try to get a better visualisation of the displacement current.



- When an alternating source, $v(t) = V_o \sin(\omega t)$, is applied to a capacitor as shown here, a time-varying conduction current, i_c, is known to flow, where

$$i_c = C\frac{dv(t)}{dt} = \omega C V_o \cos(\omega t) \qquad (15.16)$$

and, since $C \equiv \epsilon A/d$, this current has density of magnitude

$$J_c = \frac{i_c}{A} = \omega\frac{\epsilon}{d}V_o \cos(\omega t) \qquad (15.17)$$

- Now, Ampère's circuital law states that the integral of magnetic field around a closed path is equal to the current through *any* surface that is bounded by that path. Equivalently, it is equal to the normal component of current density, integrated over *any* surface bounded by the path. This should apply at any instant or over periods of time, and it must be completely independent of the surface chosen.

- Clearly, if we choose to integrate only conduction current density over surfaces S_1 and S_2 in the figure above, then we will get a non-zero result for S_1 in general, but zero for S_2, since no conduction current can be flowing in the dielectric of the capacitor.

- We note, however, that the electric field in the capacitor, which has magnitude $E = v(t)/d$, is time-varying, so that

$$\frac{\partial D}{\partial t} = \epsilon \frac{\partial E}{\partial t} \neq 0 \tag{15.18}$$

In fact, there is a *displacement current* within the capacitor whose density has magnitude

$$J_d = \frac{\partial D}{\partial t} = \epsilon \frac{\partial E}{\partial t} = \frac{\epsilon}{d}\frac{dv(t)}{dt} = \omega \frac{\epsilon}{d} V_o \cos(\omega t) \quad (= J_c) \tag{15.19}$$

Assuming that no fringing occurs outside the capacitor plates, the displacement current i_d may then be found from

$$i_d = A|\mathbf{J}_d| = \omega \left(\frac{\epsilon A}{d}\right) V_o \cos(\omega t) = \omega C V_o \cos(\omega t) \tag{15.20}$$

- But this is precisely the expression found for i_c above! Therefore, continuity of *overall* current is preserved at the capacitor's plates. In the absence of fringing (so that **D** varies *only* inside the capacitor), and since $\mathbf{J}_c = 0$ within the capacitor's dielectric, we see that ONLY i_c contributes to the integral over S_1 and ONLY i_d to the integral over S_2. This means that our requirement of consistency from Ampère's Law *is* satisfied by the new version stated in the fourth equation above, because

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J}_c \cdot d\mathbf{S} + 0 = 0 + \frac{\partial}{\partial t}\int_{S_2} \mathbf{D} \cdot d\mathbf{S} \tag{15.21}$$

and, in this example,

$$\frac{\partial \mathbf{D}}{\partial t} \equiv \mathbf{J}_d = \mathbf{J}_c \tag{15.22}$$

- We come to an understanding, therefore, that displacement current is vital to the consistency of Maxwell's equations, even though it is not a directly measurable quantity.

Chapter 16

Sinusoidally-Varying EM Fields

16.1 The use of phasors

- Time-variation of electromagnetic fields is very often either sinusoidal or else periodic, and in the latter case it is subject to Fourier decomposition into sinusoidal components.

- You may recall from your circuit analysis course that the mathematical handling of sinusoidal functions is made easier by carrying the amplitude and phase information about the sinusoid in the form of a complex number known as a *phasor*.

- Thus, the sinusoidal function $v(t) = V_m \cos(\omega t + \phi)$, which has amplitude V_m and phase ϕ with respect to some agreed reference, is represented by the phasor

  $$\tilde{\mathbf{V}} = V_m \angle\phi = V_m \cos(\phi) + jV_m \sin(\phi) = V_m e^{j\phi} \tag{16.1}$$

  Note that the frequency, ω, is assumed known.

- The time-dependent function $v(t)$, may then be recovered from the phasor representation by multiplying the phasor by $e^{j\omega t}$ and then taking the real part of the product. Thus

$$\begin{aligned} v(t) &= \Re(\tilde{\mathbf{V}}e^{j\omega t}) \\ &= \Re(V_m e^{j(\omega t+\phi)}) \\ &= \Re(V_m \cos(\omega t + \phi) + jV_m \sin(\omega t + \phi)) \\ &= V_m \cos(\omega t + \phi) \qquad (16.2) \end{aligned}$$

exactly as we defined it.

- One motivation for working with complex exponential phasors rather than with sinusoids is that it makes the differentiation of field quantities with respect to time very much easier to handle compactly. The time-derivative of a phasor is simply the phasor multiplied by $j\omega$. In this way,

$$\frac{dv(t)}{dt} = -\omega V_m \sin(\omega t + \phi) \qquad (16.3)$$

can be represented by $j\omega\tilde{\mathbf{V}}$ because

$$\begin{aligned} \frac{dv(t)}{dt} &= \Re(j\omega\tilde{\mathbf{V}}e^{j\omega t}) \\ &= \Re(j\omega V_m e^{j(\omega t+\phi)}) \\ &= \Re(j\omega V_m \cos(\omega t + \phi) + j^2\omega V_m \sin(\omega t + \phi)) \\ &= -\omega V_m \sin(\omega t + \phi) \qquad (16.4) \end{aligned}$$

after the phasor representation is returned to the time-domain form.

$$\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} E_x(x,y,z) \\ E_y(x,y,z) \\ E_z(x,y,z) \end{pmatrix} = \begin{pmatrix} \tilde{E}_x \\ \bar{E}_y \\ \bar{E}_z \end{pmatrix} = \tilde{E}$$

16.2 Maxwell's equations in phasor form

- We can now re-write Maxwell's equations for the case of sinusoidal time-variation. Note carefully that although we use the same symbols in these equations, they actually all denote complex numbers which carry the magnitude and phase information about each field quantity at a particular instant in time. To "unlock" the time-dependent function from each of these phasors, we would multiply by $e^{j\omega t}$ and take the real part, as described above.
- Maxwell's equations then become, in phasor form:
  1. $\nabla \cdot \tilde{\mathbf{D}} = \rho(x, y, z, t)$
  2. $\nabla \cdot \tilde{\mathbf{B}} = 0$
  3. $\nabla \times \tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{B}}$
  4. $\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}$
- Note the compactness of this form of expression. Equation 3, for example, can be decomposed into

$$\nabla \times \tilde{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E}_x & \tilde{E}_y & \tilde{E}_z \end{vmatrix} \tag{16.5}$$

$$= \left(\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}\right)\hat{\mathbf{x}} - \left(\frac{\partial \tilde{E}_z}{\partial x} - \frac{\partial \tilde{E}_x}{\partial z}\right)\hat{\mathbf{y}} + \left(\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$= -j\omega\tilde{B}_x\hat{\mathbf{x}} - j\omega\tilde{B}_y\hat{\mathbf{y}} - j\omega\tilde{B}_z\hat{\mathbf{z}} \tag{16.6}$$

so that, for example,

$$\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z} = -j\omega\tilde{B}_x \tag{16.7}$$

in which the quantities shown are still all *phasors* whose time-dependent equivalents can be unlocked in the usual way.

16.3 Plane waves in free space

- Fortunately, it is often possible to make massive simplifications to these sets of equations and still to be able to study some very realistic situations with them.

- Take, for instance, the behaviour of an electromagnetic wave which varies along only one axis (the z-axis, say), and which exists in a region in which there is no free charge ($\rho = 0$) and also no movement of charge ($\mathbf{J} = 0$). We call this a *plane wave* in *free space*.

- It turns out that electromagnetic waves travelling in a vacuum (e.g. through outer space) are a good example of plane waves in free space. Since the interplay of electric and magnetic fields in such waves is governed by Maxwell's equations, we are motivated to apply Maxwell's equations with $\rho = 0$, $\mathbf{J} = 0$, $\frac{\partial}{\partial x} = 0$ and $\frac{\partial}{\partial y} = 0$ to see what can be said about light, microwaves, radio waves etc. when they travel through a vacuum (or near vacuum).

- For plane waves in free space, Maxwell's equations are given below. Their phasor equivalents, which we will use later, are given in brackets.

  1. $\nabla \cdot \mathbf{D} = 0$ $\quad (\nabla \cdot \tilde{\mathbf{D}} = 0)$
  2. $\nabla \cdot \mathbf{B} = 0$ $\quad (\nabla \cdot \tilde{\mathbf{B}} = 0)$
  3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$ $\quad (\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}})$
  4. $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$ $\quad (\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon\tilde{\mathbf{E}})$

- Now, remembering that $\partial/\partial x = 0$ and $\partial/\partial y = 0$ for a plane wave, the decomposition of the equations is a bit easier. Equation 3 now gives

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{16.8}$$

Helmholtz equations $\frac{\partial \tilde{E}_x}{\partial z} = -j\omega\mu \tilde{H}_y$

so that we have the three relations

diff w.r.t. z

$$-\frac{\partial E_y}{\partial z} = -\mu\frac{\partial H_x}{\partial t}, \qquad \frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t}, \qquad 0 = -\mu\frac{\partial H_z}{\partial t} \tag{16.9}$$

$$-\frac{\partial^2 E_y}{\partial z^2} = -\mu \frac{\partial^2}{\partial t\,\partial z}$$

Similarly, equation 4 now yields

$$\nabla\times\mathbf{H} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \epsilon\frac{\partial\mathbf{E}}{\partial t} \tag{16.10}$$

w.r.t. t $\frac{\partial^2 H_x}{\partial t\,\partial z} = \epsilon\frac{\partial^2 E_y}{\partial t^2}$

$$-\frac{\partial^2 E_y}{\partial z^2} = \mu\epsilon\frac{\partial^2 E_y}{\partial t^2}$$

giving us three more equations:

$$-\frac{\partial H_y}{\partial z} = \epsilon\frac{\partial E_x}{\partial t}, \qquad \frac{\partial H_x}{\partial z} = \epsilon\frac{\partial E_y}{\partial t}, \qquad 0 = \epsilon\frac{\partial E_z}{\partial t} \tag{16.11}$$

- We can now combine the second equation from $\nabla\times\mathbf{E}$ with the first from $\nabla\times\mathbf{H}$, for example. Differentiating the second equation from $\nabla\times\mathbf{E}$ with respect to z gives

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu\frac{\partial^2 H_y}{\partial z\partial t} \tag{16.12}$$

Likewise, differentiating the first equation from $\nabla\times\mathbf{H}$ with respect to time yields

$$-\frac{\partial^2 H_y}{\partial z\partial t} = \epsilon\frac{\partial^2 E_x}{\partial t^2} \tag{16.13}$$

We combine these results by eliminating H_y to arrive at

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon\frac{\partial^2 E_x}{\partial t^2} \tag{16.14}$$

- This is a second-order partial differential equation, known as the *one-dimensional wave equation*, which we must solve if we wish to understand how E_x will perform as part of the planar free-space wave. Note that we could just as easily have combined the above equations to obtain wave equations for E_y, H_x or H_y.

16.4 Solving the 1-D wave equation

- Let us suppose that some function $F(z,t)$ satisfies the one-dimensional wave equation, so that we may write

$$\frac{\partial^2 F}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} \tag{16.15}$$

  where v is a constant whose meaning and value are to be determined shortly.

- Mathematical solutions to this equation take the form

$$F(z,t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) \tag{16.16}$$

$$\frac{dF}{dz} = -\frac{1}{v} f'\left(t - \frac{z}{v}\right) + \frac{1}{v} f'\left(t + \frac{z}{v}\right)$$

$$\frac{\partial^2 F}{\partial z^2} = \frac{1}{v^2}\left(f''\left(t - \frac{z}{v}\right) + f''(t + z/v)\right)$$

  where f_1 is some function of $(t - z/v)$ and f_2 is a function of $(t + z/v)$, so all the brackets here denote arguments of the functions (not multiplication!). It is easy to see that this is a form that a solution could take. Simply re-substitute it into both sides of the wave equation (you should do this for yourself as a check).

- Now think about $f_1(t - z/v)$, a function that varies in both space (z) and time (t). Imagine the value of this function at $z = 0$ and $t = 0$. A moment later, t has moved on, so $(t - z/v)$ will have changed, and so the value of the function at $z = 0$ will now be different. If we wanted to "stay with" the same value that we had at $z = 0$ and $t = 0$, evidently we should have to keep $(t - z/v)$ constant. But if

$$t - \frac{z}{v} = K \quad \text{(a constant)} \tag{16.17}$$

  then

$$z = vt - Kv \tag{16.18}$$

  and we see that we should have to move along the z-axis with velocity $dz/dt = v$ in order to "stay with the same point" on the f_1 function. This is tantamount to saying that $f_1(t - z/v)$ *moves along the z-axis with velocity* v.

- By a very similar argument, $f_2(t+z/v)$ can be shown to move *backwards* along the z-axis, with velocity $-v$. Check that you can visualise this.

16.5 Velocity of electromagnetic propagation

- We can now apply all of this theory to plane electromagnetic waves in free space, travelling in the z-direction. We know that, in this case, E_x satisfies the wave equation

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon\frac{\partial^2 E_x}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 E_x}{\partial t^2} \tag{16.19}$$

- Most commonly, electromagnetic waves have sinusoidal time-variation, so that the solution to the wave equation, which is of the form

$$E_x = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) \tag{16.20}$$

  has sinusoidal components f_1 and f_2.

- This suggests the use of phasors to solve the wave equation, which can be written for this purpose as the *one-dimensional Helmholtz equation*

$$\frac{d^2 \tilde{E}_x}{dz^2} = -\omega^2\mu\epsilon\tilde{E}_x \tag{16.21}$$

  simply by replacing time differentiation with multiplication by $j\omega$, and then recognising that $\tilde{E}_x$ is now a function of one variable only (z), so that the partial derivative can be replaced by a total derivative.

- Because $\tilde{E}_x$ is a phasor in complex exponential form, the solution components f_1 and f_2 should also be written as complex exponentials, and are also functions of z only. Hence

$$\tilde{E}_x = c_1 e^{-jkz} + c_2 e^{jkz} \tag{16.22}$$

  where $k = \omega\sqrt{\mu\epsilon}$. You should verify, by re-substitution, that this solution does indeed satisfy the Helmholtz equation.

- We can now recover the time-varying solution for $E_x(t)$ from the phasor $\tilde{E}_x$ by simply writing

$$\begin{aligned}E_x(z,t) &= \Re(\tilde{E}_x e^{j\omega t}) \\ &= \Re(c_1 e^{-jkz} e^{j\omega t} + c_2 e^{jkz} e^{j\omega t}) \\ &= \Re(c_1 e^{j(\omega t - kz)} + c_2 e^{j(\omega t + kz)}) \\ &= c_1 \cos(\omega t - kz) + c_2 \cos(\omega t + kz) \qquad (16.23)\end{aligned}$$

which is, exactly as expected, a pair of sinusoids, each of frequency ω, and travelling in the $+z$ and $-z$ directions respectively.

- The *velocity* of these waves is found, by comparison with the wave equation, to be $v = 1/\sqrt{\mu\epsilon}$, and in a vacuum this takes the value

$$\begin{aligned}v &= \frac{1}{\sqrt{\mu_o \epsilon_o}} \qquad (16.24) \\ &= \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}}} \\ &\approx 2.9979 \times 10^8 \quad [\mathrm{ms}^{-1}]\end{aligned}$$

We therefore have the very satisfactory result that light (or any other electromagnetic wave) travels at the velocity expected from field theory. We can also predict the velocity of electromagnetic propagation in any other medium, if we know the values of $\epsilon = \epsilon_r \epsilon_o$ and $\mu = \mu_r \mu_o$ in that medium.

- Note, finally, that k, which is known as the *angular wave number* of the travelling waveform, signifies the number of complete waves in 2π metres along the z-axis and can be written

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{v} \qquad (16.25)$$

and that therefore the distance between similar points in a periodic wave, which is given by $2\pi/k$, is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi v}{\omega} = \frac{v}{f} \qquad (16.26)$$

where $f = \omega/2\pi$ is known as the *frequency* and λ is the *wavelength* of the periodic waveform.

Chapter 17

The Skin-Depth Phenomenon

17.1 Maxwell's equations in good conductors

- The previous chapter dealt with Maxwell's equations in free space—an excellent dielectric. We now study the behaviour of electromagnetic fields in the vicinity of a good conductor. The *conductivity* of such a medium is given at frequencies up to many gigahertz, by the real number σ, (although it may be noted that σ is complex at optical frequencies). We can therefore state the familiar field-theory version of Ohm's Law in phasor form:

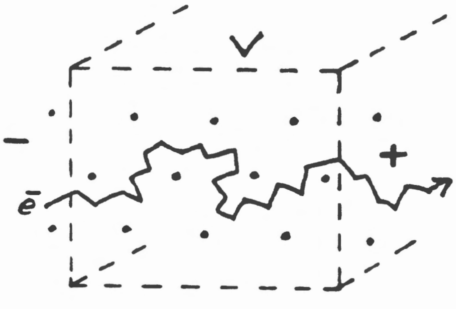
$$\tilde{\mathbf{J}} = \sigma \tilde{\mathbf{E}} \tag{17.1}$$

- Now since $\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\epsilon\tilde{\mathbf{E}}$ (the 4th of Maxwell's equations), a convenient factorisation becomes possible, and so we can write

$$\nabla \times \tilde{\mathbf{H}} = (\sigma + j\omega\epsilon)\tilde{\mathbf{E}} \tag{17.2}$$

  It is important to note that in good conductors $\sigma \gg \omega\epsilon$ in the bracketed term above, while in good dielectrics **J** is insignificant and we have been able to assume that $\omega\epsilon \gg \sigma$, in free space

- We have a picture of how conduction takes place in a conductor through the drifting of charge carriers within a lattice of atoms when a field is applied across the lattice.



Carriers are promoted from a *valence* to a *conduction* rôle in the material by the addition of energy in the form of heat or light. The total number of carriers (electrons) in a given volume V equals the total number of positively-charged nucleons, however. As long as V is large with respect to individual atoms yet small in comparison with major circuit elements, the *net charge* within V will therefore be zero. This picture is in good agreement with our field theory, as can be seen by taking the divergence of both sides of the equation above:

$$\nabla \cdot \nabla \times \tilde{\mathbf{H}} = (\sigma + j\omega\epsilon)\nabla \cdot \tilde{\mathbf{E}} \tag{17.3}$$

Since $\nabla \cdot \nabla \times \tilde{\mathbf{H}} \equiv 0$ by definition, we arrive at $\nabla \cdot \tilde{\mathbf{E}} = 0$ and so

$0 = \nabla\cdot\tilde{E} = \frac{1}{\varepsilon}\nabla\cdot\tilde{D} = \nabla\cdot\tilde{D} = \rho = 0$

$$\nabla \cdot \tilde{\mathbf{D}} = \rho = 0 \tag{17.4}$$

confirming that the net charge, ρ, will be zero when Ohm's Law applies.

17.2 $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{J}}$ in good conductors

$\nabla\times\nabla\times\vec{F} = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$

Do it!

- It is quite straightforward to find differential equations which describe the behaviour of field quantities $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{J}}$ in the vicinity of a large, good conductor. The general pattern for deriving these relationships is to work with Maxwell's equations, Ohm's Law and the vector identity $\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$ that we met earlier in the course. You ought to gain confidence in *all three* of the derivations below.

I: In the case of the electric field intensity, $\tilde{\mathbf{E}}$, we can begin with the third Maxwell equation, $\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$, and then take the curl of both sides to get,

$$\nabla \times \nabla \times \tilde{\mathbf{E}} \equiv \nabla(\nabla \cdot \tilde{\mathbf{E}}) - \nabla^2\tilde{\mathbf{E}} = -j\omega\mu\nabla \times \tilde{\mathbf{H}} \tag{17.5}$$

Then, since Ohm's Law leads to $\nabla \cdot \tilde{\mathbf{E}} = 0$ (as we saw above) we obtain

$$\nabla^2\tilde{\mathbf{E}} = j\omega\mu(\sigma + j\omega\epsilon)\tilde{\mathbf{E}} \tag{17.6}$$

and you will notice that we have also replaced $\nabla \times \tilde{\mathbf{H}}$ with the Ohm's Law version that we derived above. Now in a conductor $\sigma \gg \omega\epsilon$, and so we can effectively neglect displacement current altogether and write

$$\nabla^2\tilde{\mathbf{E}} = j\omega\mu\sigma\tilde{\mathbf{E}} \tag{17.7}$$

$\nabla\cdot(\hat{A}\times\hat{B}) = \tilde{B}(\nabla\times\hat{A}) - \tilde{A}(\nabla\times\hat{B})$

II: Similarly, for the magnetic field intensity, $\tilde{\mathbf{H}}$, we might begin with

$$\nabla \times \tilde{\mathbf{H}} = (\sigma + j\omega\epsilon)\tilde{\mathbf{E}} \approx \sigma\tilde{\mathbf{E}} \tag{17.8}$$

Taking the curl of both sides now gives

$$\nabla \times \nabla \times \tilde{\mathbf{H}} = \nabla(\nabla \cdot \tilde{\mathbf{H}}) - \nabla^2\tilde{\mathbf{H}} = \sigma\nabla \times \tilde{\mathbf{E}} \tag{17.9}$$

But $\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$ by the third Maxwell phasor equation, and $\nabla \cdot \tilde{\mathbf{H}} = 0$ by the second, and so

$\downarrow \nabla \cdot \frac{\vec{B}}{\mu}$

$$\nabla^2\tilde{\mathbf{H}} = +j\omega\mu\sigma\tilde{\mathbf{H}} \tag{17.10}$$

III: Finally, for the conduction current density, $\tilde{\mathbf{J}}$, a possible starting point is to take the curl of both sides of Ohm's Law and then to apply the third Maxwell equation, $\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$, to obtain:

$\vec{J} = \sigma\vec{E}$

$$\nabla \times \tilde{\mathbf{J}} = \sigma\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\sigma\tilde{\mathbf{H}} \tag{17.11}$$

Next, we can take the curl of both sides again, and this time replace $\nabla \times \tilde{\mathbf{H}}$ to get

$$\nabla \times \nabla \times \tilde{\mathbf{J}} = -j\omega\mu\sigma\nabla \times \tilde{\mathbf{H}} = -j\omega\mu\sigma(\sigma + j\omega\epsilon)\tilde{\mathbf{E}} \tag{17.12}$$

Now, remembering that $\sigma \gg \omega\epsilon$ for a conductor, we can neglect the displacement current term $j\omega\epsilon\tilde{\mathbf{E}}$ and replace the other term, $\sigma\tilde{\mathbf{E}}$, with $\tilde{\mathbf{J}}$ (by Ohm's Law) to get

$$\nabla \times \nabla \times \tilde{\mathbf{J}} = \nabla(\nabla \cdot \tilde{\mathbf{J}}) - \nabla^2\tilde{\mathbf{J}} = -j\omega\mu\sigma\tilde{\mathbf{J}} \tag{17.13}$$

Then, finally, we must appeal to the concept of *continuity*, in which we saw that $\nabla \cdot \tilde{\mathbf{J}} = -\frac{\partial \rho}{\partial t} \approx 0$ for a good conductor in the steady state. This then leads us to

$\nabla \cdot J = -\frac{d\rho}{dt} = 0$

$$\nabla^2\tilde{\mathbf{J}} = j\omega\mu\sigma\tilde{\mathbf{J}} \tag{17.14}$$

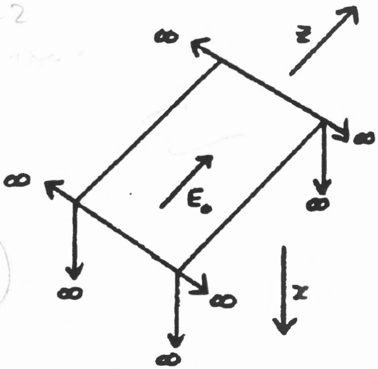
- The differential equations governing $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{J}}$ are therefore of identical form in a good conductor. We now turn our attention to solving for $\tilde{\mathbf{E}}$: the other solutions follow the same pattern.

17.3 Field penetration into a good conductor

- The following analysis will show that the $\tilde{\mathbf{E}}$-field (and, similarly $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{J}}$) only penetrates with significant amplitude to a limited depth into a good conductor. At high frequencies, it will be found that this depth is small in comparison to the wavelength. This implies that the performance of high frequency fields near a conductor is not affected much by the shape, edges or curvature of the conductor, because typical conductor dimensions are a lot higher than the penetration depth of the field. We therefore study an $\tilde{\mathbf{E}}$-field of magnitude E_o at the surface of a conductor, and can make the assumption that the conductor is infinite in extent below the surface.

$$\nabla^2\tilde{E} = \hat{x}\,\nabla^2\tilde{E}_x + \hat{y}\,\nabla^2\tilde{E}_y + \hat{z}\,\nabla^2\tilde{E}_z$$

$$= \hat{x}\left(\frac{\partial^2 \tilde{E}_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}\right) + \hat{y}\left(\frac{\partial^2 \tilde{E}_y}{\partial x^2} + \frac{\partial^2 \tilde{E}_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2}\right) + \hat{z}\left(\frac{\partial^2 \tilde{E}_z}{\partial x^2} + \frac{\partial^2 \tilde{E}_z}{\partial y^2} + \frac{\partial^2 \tilde{E}_z}{\partial z^2}\right)$$



$\vec{E} \wedge \vec{H} = \bigcirc\!\!\!\!x$ direction of propagation

Poynting's [illegible]

- We also assume that the $\tilde{\mathbf{E}}$-field is a plane wave, which is directed in the z-direction and which propagates in the x-direction. There is therefore no variation of electric field with y or z. As this means that $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$, our differential equation for $\tilde{\mathbf{E}}$, namely $\nabla^2\tilde{\mathbf{E}} = j\omega\mu\sigma\tilde{\mathbf{E}}$, becomes

$E_x, E_y = 0$

$> \frac{\partial}{\partial y} = \frac{\partial}{\partial z}$

$$\frac{d^2\tilde{E}_z}{dx^2} = j\omega\mu\sigma\tilde{E}_z = \tau^2\tilde{E}_z \quad \hat{z} \tag{17.15}$$

for which

$$\tau = \sqrt{j\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{2\pi f\mu\sigma} = \frac{1+j}{\delta} \tag{17.16}$$

where the factor $\frac{1+j}{\sqrt{2}}$ is recognised as the positive square root of j, f is the frequency in hertz, and $\delta \equiv \frac{1}{\sqrt{\pi f\mu\sigma}}$ in metres.

$\vec{F}\cdot d\vec{l} = dq(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{v}\,dt = \vec{E}\cdot\vec{v}\,dq\,dt$

- The differential equation for $\tilde{E}_z$ can now be written

$$\frac{d^2\tilde{E}_z}{dx^2} - \tau^2\tilde{E}_z = 0 \tag{17.17}$$

and its characteristic equation, $s^2 - \tau^2 = 0$, implies that $(s+\tau)(s-\tau) = 0$ and so $s = \pm\tau$. The solution is therefore

$$\tilde{E}_z = Ae^{-\tau x} + Be^{\tau x} \tag{17.18}$$

where A and B are constants to be determined from the boundary conditions. We quickly see that, since $\tilde{E}_z$ certainly cannot increase to infinity as x increases, B must equal zero and since, when $x = 0$, $\tilde{E}_z = E_o$, A must equal E_o. Hence our solution for the differential equation is the decaying exponential

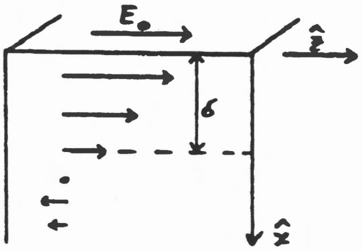
magnitude (envelope)

$$\tilde{E}_z = E_o e^{-\tau x} = E_o e^{-\left(\frac{1+j}{\delta}\right)x} = E_o e^{-\frac{x}{\delta}} e^{-j\frac{x}{\delta}} \tag{17.19}$$

oscillates

- Likewise, you could arrive at $\tilde{H}_y = H_o e^{-\frac{x}{\delta}} e^{-j\frac{x}{\delta}}$ or $\tilde{J}_z = J_o e^{-\frac{x}{\delta}} e^{-j\frac{x}{\delta}}$ by very similar developments.

- The interpretation of the $\tilde{\mathbf{E}}$-field solution is shown in this figure. You will see that the magnitude of the $\hat{\mathbf{z}}$-component of $\tilde{\mathbf{E}}$ is E_o/e at depth $x = \delta$, which is known as the *skin depth*, and that the envelope of the magnitude of the $\tilde{\mathbf{E}}$-field decays exponentially with depth. Furthermore, the decaying sinusoidal wave associated with the $\tilde{\mathbf{E}}$-field travels into the conductor in the $\hat{\mathbf{x}}$-direction.



- Note also that, in all cases the $\mathbf{E}$ (or $\mathbf{D}$) and $\mathbf{H}$ (or $\mathbf{B}$) fields have direction which is *uniquely* determined by the right-hand set $[\mathbf{E}, \mathbf{H}, \hat{\mathbf{x}}]$, where $\hat{\mathbf{x}}$ is in the direction of propagation of the wave, and $\mathbf{E} = [E_x, E_y, E_z]^T$.

17.4 Resistance and internal impedance

resistivity.

$R = \frac{R_s' L}{A}$

- Let us now consider a section of the semi-infinite plane of unit width and length (but effectively infinite depth). We can calculate the total current flowing in this conductor by integrating the exponentially-decaying current density down to infinite depth. Thus

$$I_{tot} = \int_0^\infty J_o e^{-\left(\frac{1+j}{\delta}\right)x}\, dx = \frac{J_o \delta}{1+j} \tag{17.20}$$

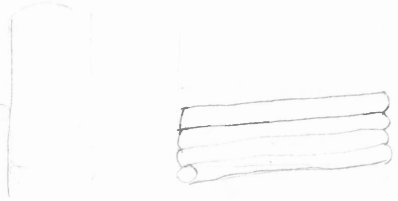
- The magnitude of the surface electric field is then simply found from the surface current density to be:

$$E_o = \frac{J_o}{\sigma} \tag{17.21}$$

The *internal impedance* of the conductor per unit length and width is the ratio of the field and of the current at the surface, which is

$$Z_s = R_s + j\omega L_i = \frac{E_o}{I_{tot}} = \frac{1}{\sigma\delta} + \frac{j}{\sigma\delta} \tag{17.22}$$

-
    - Note that this expression gives us the *surface resistivity*, R_s, of the conductor (*multiply* it by length and *divide* by width to get the *resistance* of the conductor), and also the *internal reactance*, $j\omega L_i$. The latter does not include any contribution to total inductance from fields that are external to the conductor.
    - Note also that for a plane conductor of unit length and width, the resistance and internal reactance are of *equal magnitude* or, equivalently, that the phase angle of the internal impedance, Z_s, is 45°.
    - Finally, note that the resistance of the conductor is inversely proportional to skin depth, δ, which is itself inversely proportional to the square root of the frequency, f. Thus, at high frequencies we understand that fields penetrate less deeply into a conductor, and that the resistance of the conductor *increases*. This is in line with our expectation for a conductor of decreased cross-sectional area.



Chapter 18

The Basis of Kirchhoff's Laws

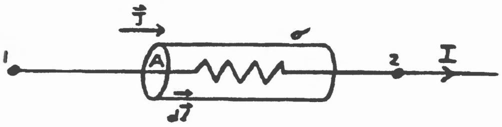
18.1 Lumped elements

- In everyday electrical engineering, the great majority of electromagnetic interactions are analysed and evaluated using standard *circuit theory*. The purpose of this chapter is to show that all circuit theory has its foundations in field theory, and that therefore the use of circuit analysis techniques is justified upon solid theoretical grounds.

- In lumped-element circuit analysis, we make the assumption that all of the circuit elements used are small compared with the wavelength of the electromagnetic fields within them. This assumption allows us to treat the conditions around each element as *quasistatic*, simplifying our treatment considerably. At high frequencies (e.g. in microwave circuits) or where very large circuit elements are being considered (e.g. an inter-city power transmission line), the quasistatic assumption is invalid. In such cases, it is necessary to resort to transmission-line theory in preference to standard circuit analysis. We concern ourselves here only with the validity of the latter at sufficiently low frequencies.

- The three ideal lumped elements most frequently dealt with by circuit analysis (see Module A of EEE221W) are the *resistor*, the *capacitor* and the *inductor*. We now consider each of these in turn from the standpoint of field theory.

18.2 Resistors



- The resistor is a *dissipative* element: that is, the interactions which occur within it at a subatomic level cause it always to absorb net electrical power from its surroundings. It converts much of the energy that it absorbs to the generally unusable form of heat.

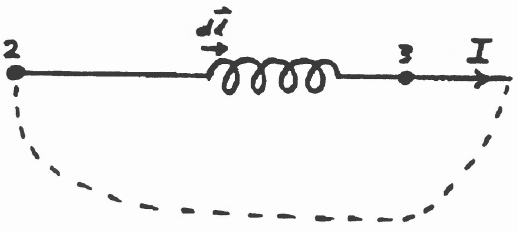
- Resistors conduct current, and are characterised by a *conductivity*, σ in ℧m^{-1}. Hence, as is well-known, they obey Ohm's Law, so that if any field $\mathbf{E}$ is applied across a resistor, then a current of density $\mathbf{J}$ will flow, where
$$\mathbf{J} = \sigma \mathbf{E} \tag{18.1}$$

- The voltage across the resistor can be expressed using the simple relation for electrostatic potential difference:
$$V_{21} = \Phi_2 - \Phi_1 = -\int_1^2 \mathbf{E} \cdot d\mathbf{l} = -\int_1^2 \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} = -\int_1^2 \frac{I\,dl}{\sigma\,A} \tag{18.2}$$
where it is clear that $\mathbf{J} \parallel d\mathbf{l}$, that $\mathbf{J} = I/A$, and that A represents the cross-sectional area of the resistor.

- If we now assume that the current, I, the conductivity, σ and the cross-sectional area, A, are all constant between points 1 and 2, we can write
$$V_{21} = -\frac{I}{\sigma\,A}\int_1^2 dl = -\frac{IL}{\sigma\,A} = -IR \tag{18.3}$$
where L is the length of the resistor, and $R = L/(\sigma\,A)$ is known as the *resistance*. We have thus recovered the familiar form of Ohm's Law.

- In the case where I, σ and/or A are not constant over the length of the resistor, we simply define
$$R = \int_1^2 \frac{dl}{\sigma\,A} \tag{18.4}$$
to arrive at
$$v_{21}(t) = -Ri(t) \tag{18.5}$$

18.3 Inductors



- Inductors, as we know, *store* energy in the magnetic field that is associated with the current that passes through them. Ideal inductors have zero resistance in their coils, and so all of the stored energy may be recovered at a later time.

- Taken around the path shown in the diagram, clockwise from point 2, the integral of the electric field around the path is evidently

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_2^3 \mathbf{E} \cdot d\mathbf{l} + \int_3^2 \mathbf{E} \cdot d\mathbf{l} \tag{18.6}$$

where the first term on the right-hand side integrates the field along the perfectly-conducting coil, and is therefore equal to zero, while the second term integrates the field along the return path between the terminals. We thus have, by Faraday's Law,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 + \int_3^2 \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{18.7}$$

- We defined *inductance* much earlier in the course as

$$L = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{18.8}$$

so we may now write, for the quasistatic case,

$$v_{23}(t) = -\int_3^2 \mathbf{E} \cdot d\mathbf{l} = \frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} = \frac{\partial}{\partial t}(Li(t)) = L\frac{di(t)}{dt} \tag{18.9}$$

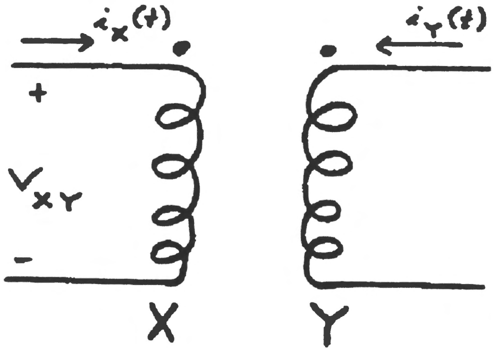
if the current through the inductor is taken to be the time-varying $i(t)$. Again, this is a very familiar result from circuit theory.

- Note that if the resistance of the coil were not zero, but instead had value R_L, we would then have

$$\oint \mathbf{E} \cdot d\mathbf{l} = +R_L i(t) - v_{23}(t) = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{18.10}$$

so that, as we might expect, a time-varying current would now give us

$$v_{23}(t) = R_L i(t) + L\frac{di(t)}{dt} \tag{18.11}$$



- Note also that a voltage may appear across an inductive coil in a circuit as a result of magnetic flux changes that are due to some *other* source of magnetic flux. Thus, if coil Y lies close to coil X and has a time-varying current $i_Y(t)$ flowing in it, then the magnetic flux that links coil X due to this current is written ψ_{XY}. By direct analogy with the concept of self-inductance, L, the *mutual inductance*, M, of the coils is defined as the ratio of the flux linkage and the current in coil Y that produces it. Thus

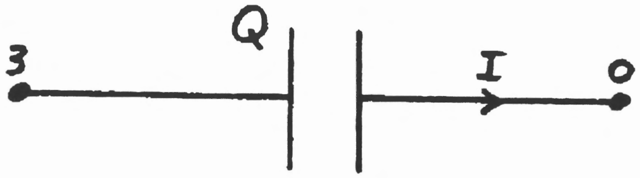
$$M_{XY} = \frac{\psi_{XY}}{i_Y} \tag{18.12}$$

The voltage induced across coil X by this flux linkage is then written

$$v_{XY}(t) = \frac{d\psi_{XY}}{dt} = M_{XY}\frac{di_Y(t)}{dt} \tag{18.13}$$

- The phenomenon of mutual inductance is symmetrical in the sense that the same value for M is used in calculating the voltage in coil Y due to changes in the current in coil X. Note that dots drawn at the ends of the coils in the circuit diagram show the directions of the windings, and so give the polarity sign for the voltage induced.

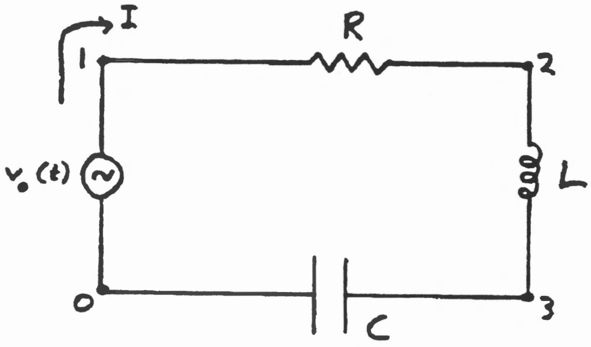
18.4 Capacitors



- Capacitors store energy in the electric fields between their plates, and we established much earlier on that, with a charge Q on either plate, the capacitance is found using the potential difference between the plates, as
$$C = \frac{Q}{\Phi_3 - \Phi_0} = \frac{Q}{v_{30}(t)} \tag{18.14}$$
- Any current flowing in the capacitor will, by continuity, merely raise or lower the value of Q. Thus we can write
$$i(t) = \frac{dQ}{dt} = \frac{\partial}{\partial t}(Cv_{30}(t)) = C\frac{dv_{30}(t)}{dt} \tag{18.15}$$
or, equivalently,
$$v_{30}(t) = \frac{1}{C}\int_0^t i(t)\,dt \tag{18.16}$$
if the capacitance is constant over time, and if the capacitor is initially uncharged. We have therefore recovered the familiar expressions for the behaviour of an ideal capacitor in a low-frequency circuit.

18.5 Kirchhoff's Voltage Law

- We can now put together the three elements considered in this chapter, and drive them with a time-varying voltage source, as shown in the diagram below.



Taking the line integral of electric field around the whole path, clockwise from point 0, now gives us

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_0^1 \mathbf{E} \cdot d\mathbf{l} + \int_1^2 \mathbf{E} \cdot d\mathbf{l} + \int_2^3 \mathbf{E} \cdot d\mathbf{l} + \int_3^0 \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{18.17}$$

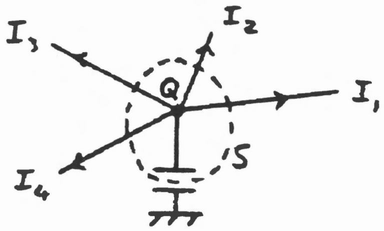
- The surface S referred to in the equation above is, of course, any surface bounded by the loop of the circuit. If we assume that the magnetic flux in the inductor is largely confined to the inside of the coil (a fair assumption, as we know from our study of the solenoid), we can see that the component of stray magnetic flux perpendicular to the surface S is negligible. Insofar as it needs to be considered at all, it could in any case be added to the effect of the inductor, L. Hence, the term on the right-hand side of the above equation can be regarded as equalling zero. The equation thus becomes

$$\begin{aligned} -\oint \mathbf{E} \cdot d\mathbf{l} &= V_{10} + v_{21}(t) + v_{32}(t) + v_{03}(t) && (18.18) \\ &= v_o(t) - Ri(t) - L\frac{di(t)}{dt} - \frac{1}{C}\int_0^t i(t)\,dt && (18.19) \\ &= 0 && (18.20) \end{aligned}$$

- This is what we know as *Kirchhoff's Voltage Law*, which we use habitually in electrical circuit analysis. You can now do so with confidence, knowing that it is a result which is formally derivable from electromagnetic field theory. The general statement of the law for N elements around a closed path in a circuit is

$$\sum_{n=1}^{N} V_n = 0 \tag{18.21}$$

18.6 Kirchhoff's Current Law



- Consider a node in an ideal circuit at which a number of conductors (assumed perfect) all meet. To be completely general, let us suppose that there is a charge Q at the node: evidently, this charge must be maintained by a capacitive effect, or else it would instantly be dissipated in the form of currents along the perfect conductors.

- We know that the principle of continuity states that

  $$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad \text{or} \quad \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_V \rho \, dV \tag{18.22}$$

  and we can give meaning to these expressions in this situation by imagining setting a surface S around the node, enclosing volume V, which would then have charge density $\rho = Q/V$ as shown above. In the integral form of the continuity expression,

  $$\oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{\partial}{\partial t} \int_V \rho \, dV = 0 \tag{18.23}$$

  the first term clearly represents the sum of the currents, I_1, I_2, I_3, $I_4 \ldots$ which flow in the perfect conductors. The second term is dQ/dt, the current leaving the node via the capacitor. This current is not included in the first term, since $\mathbf{J}$ denotes conduction or convection current only, and the capacitor current is a displacement current.

- Nevertheless, the equation shows that the sum of all the currents of all types out of the node is zero. We know this as *Kirchhoff's Current Law*, and we use it implicitly on a daily basis when we work with circuits. In general form, it may be written

  $$\sum_{n=1}^{N} I_n = 0 \tag{18.24}$$

- With Kirchhoff's Laws established from field theory, you may recall (see EEE221W) that the whole structure of circuit theory now follows from them. We therefore have the theoretical basis upon which to analyse circuits and their equivalents, thanks to the theory of electromagnetic fields. To be specific, Kirchhoff's two laws are necessary to the proofs of the following well-known circuit analysis techniques:
    - Nodal analysis
    - Mesh analysis
    - The theorems of Thévenin and Norton
    - AC phasor analysis

Chapter 19

Reflection of Travelling Plane Waves

19.1 Intrinsic impedance of a medium

- In introducing the concept of plane electromagnetic waves in free space, we saw how Maxwell's equations reduced (with $\mathbf{J} = \rho = 0$) to

  $$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{and} \quad \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{19.1}$$

  We can extract component expressions from these equations, and, for a plane EM wave propagating in the $\hat{\mathbf{z}}$-direction, these are:

  $$\frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t} \qquad \frac{\partial H_y}{\partial z} = -\epsilon \frac{\partial E_x}{\partial t} \tag{19.2}$$

  $$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \qquad \frac{\partial H_x}{\partial z} = \epsilon \frac{\partial E_y}{\partial t} \tag{19.3}$$

  $$0 = \mu \frac{\partial H_z}{\partial t} \qquad 0 = \epsilon \frac{\partial E_z}{\partial t} \tag{19.4}$$

  Further study of these equations quickly led us to the conclusion that the $\mathbf{E}$ and $\mathbf{H}$ fields of a travelling wave are *transverse* to the $\hat{\mathbf{z}}$-direction in which the wave propagates, and also led us to the wave equation, which we are now able to solve for any of E_x, E_y, H_x or H_y.

- We then found, for example, that the general solution to the wave equation for E_x gave us a wave travelling in the direction of $+\hat{\mathbf{z}}$ (which we will now denote E_{x+}), and also a wave of the same velocity propagating along $-\hat{\mathbf{z}}$ (which we can similarly call E_{x-}). Thus the solution was

$$E_x = E_{x+} + E_{x-} = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) \tag{19.5}$$

  where $v = 1/\sqrt{\mu\epsilon}$ was the velocity of the wave, which equals the vacuum speed of light, c, in free space.

- In the commonly-encountered situation where the time-variation of the fields is sinusoidal (with frequency ω), we used phasors and solved the Helmholtz equation to arrive at the similar solution for $\tilde{E}_x$:

$$\tilde{E}_x = \tilde{E}_{x+} + \tilde{E}_{x-} = c_1 e^{-jkz} + c_2 e^{jkz} \tag{19.6}$$

  where

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{v} \tag{19.7}$$

- Now, one of our component equations above states that

$$\frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t} \tag{19.8}$$

  which, in phasor form, becomes

$$\frac{\partial \tilde{E}_x}{\partial z} = -\mu j\omega \tilde{H}_y \tag{19.9}$$

  so that

$$\begin{aligned} \tilde{H}_y &= -\frac{1}{j\omega\mu}\frac{d\tilde{E}_x}{dz} && (19.10) \\ &= \frac{k}{\omega\mu}\left[c_1 e^{-jkz} - c_2 e^{jkz}\right] && (19.11) \\ &= \sqrt{\frac{\epsilon}{\mu}}\left[\tilde{E}_{x+} - \tilde{E}_{x-}\right] && (19.12) \end{aligned}$$

- Hence we can write $\tilde{H}_y$ as

$$\tilde{H}_y = \tilde{H}_{y+} + \tilde{H}_{y-} = \frac{\tilde{E}_{x+}}{\eta} - \frac{\tilde{E}_{x-}}{\eta} \tag{19.13}$$

where $\eta = \sqrt{(\mu/\epsilon)}$ is therefore the ratio of E_x (the x-component of **E**-field) and H_y (the y-component of **H**-field) everywhere along the wave. You can see that η is a constant for any given medium, and, because it has dimensions of (V/m)/(A/m)= Ω, it is known as the *intrinsic impedance* of the medium in which the wave propagates.

- In particular, the *impedance of free space* is

$$\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} = 376.73 \approx 120\pi \quad \Omega \tag{19.14}$$

This is a quantity with which you will frequently meet in any analysis of waves travelling in space or in similar media.

- As we remarked when studying the wave equation, it is a simple procedure to extract from Maxwell's equations the wave equation for any of the other quantities $\tilde{E}_y$, $\tilde{H}_x$ or $\tilde{H}_y$. You should practise doing this. You will then find that, if you have the solution for $\tilde{E}_y$ such that

$$\tilde{E}_y = \tilde{E}_{y+} + \tilde{E}_{y-} = c_3 e^{-jkz} + c_4 e^{jkz} \tag{19.15}$$

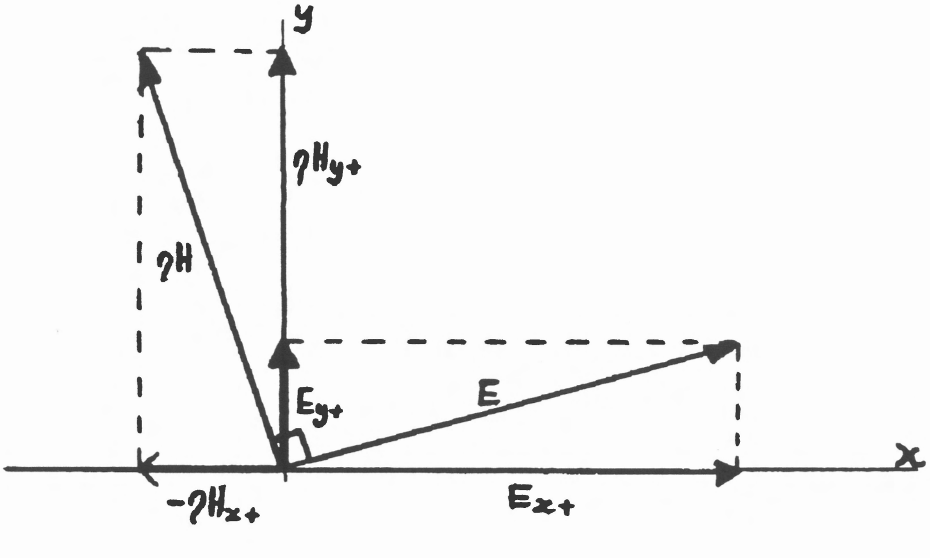
then a resubstitution into a component equation, similar to the one described above for $\tilde{E}_x$, will give you

$$\tilde{H}_x = \tilde{H}_{x+} + \tilde{H}_{x-} = -\frac{\tilde{E}_{y+}}{\eta} + \frac{\tilde{E}_{y-}}{\eta} \tag{19.16}$$

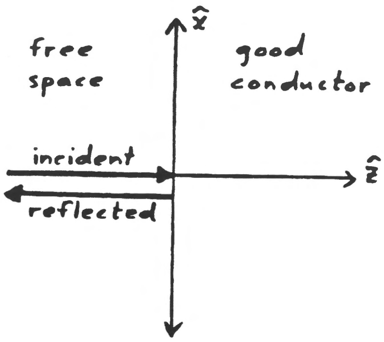
- Combining the results of both procedures, we can now see that

$$\frac{\tilde{E}_{x+}}{\tilde{H}_{y+}} = -\frac{\tilde{E}_{y+}}{\tilde{H}_{x+}} = \eta \quad \text{and} \quad \frac{\tilde{E}_{x-}}{\tilde{H}_{y-}} = -\frac{\tilde{E}_{y-}}{\tilde{H}_{x-}} = -\eta \tag{19.17}$$

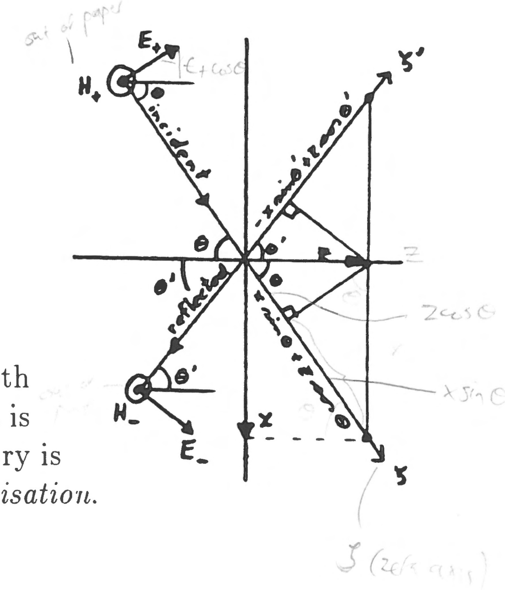
One important interpretation of these equations is that they show that the total **E** and **H** fields of a travelling plane wave are *everywhere perpendicular to one another.* This is illustrated in the following diagram, which depicts an electromagnetic wave moving out of the page (in the positive $\hat{\mathbf{z}}$-direction). Once the information contained in equation 19.17 has been included in the diagram, you can see by a similar-triangles approach that **E** and **H** must be perpendicular to each other, and also that the wave travels in the direction of $\mathbf{E} \times \mathbf{H}$, as expected.



19.2 Incidence on a perfect conductor



- We now imagine a plane wave, that has been propagating in a medium such as free space, impinging upon a perfectly-conducting surface. In the simplest case, we can visualise the wave striking the conductor so that its direction of propagation is *normal* to the conductor's surface (i.e. it is in the $\hat{\mathbf{z}}$-direction as drawn). Now, at the surface of the perfect conductor there can be no x- or y- components of **E**-field, because such a field would instantly collapse as charge carriers moved in response to it. Hence, the *negatively-travelling* part of the plane wave must, at the surface, possess x- and y- components of **E**-field which exactly balance (i.e. they are equal and opposite to) the components of the impinging, positively-travelling part of the wave. Under these circumstances, we would refer to the negatively-travelling part of the wave as a *reflection*; and, since its components are equal in magnitude but opposite in direction to those of the incident wave, it is clear that the reflected wave is very similar to the incident wave, but that it moves *away from* the conducting surface, as expected.



- Next imagine a plane wave impinging upon the conductor at some angle θ to the normal line as shown here. Again, the x- and y- components of **E**-field at the conductor's surface must each total zero, and so we expect there to be a reflected travelling plane wave, which is drawn here at an angle of θ' to the normal.

- In this example, we have drawn the **H**-field coming out of the paper, and the **E**-field lies in the *plane of incidence*, which is the plane containing both the direction of propagation and the normal line (it is the plane of the paper as drawn here). This geometry is referred to as *transverse magnetic*, or *parallel*, *polarisation*.

Note how the diagram preserves the perpendicularity of **E** and **H**, and note also the fact that $\mathbf{E} \times \mathbf{H}$ gives the direction of propagation for both the incident and the reflected waves.

- Now, the directions of the incident and reflected waves have been denoted ζ and $(-)\zeta'$ respectively, so we can write the whole **E**-field of the wave as .

$$\mathbf{E} = E_+ e^{-jk\zeta} + E_- e^{jk\zeta'} \tag{19.18}$$

The diagram also shows how we can write ζ and ζ' in terms of x and z as follows:

$$\zeta = x \sin\theta + z\cos\theta \quad \text{and} \quad \zeta' = -x\sin\theta' + z\cos\theta' \tag{19.19}$$

Hence, the whole **E**-field of the wave may be given as

$$\mathbf{E}(x,z) = E_+ e^{-jk(x\sin\theta + z\cos\theta)} + E_- e^{jk(-x\sin\theta' + z\cos\theta')} \tag{19.20}$$

Just considering the x-component of **E**-field, and still with reference to the diagram, you can easily see that taking x-components gives

$$E_x(x,z) = -E_+ \cos\theta e^{-jk(x\sin\theta + z\cos\theta)} + E_- \cos\theta' e^{jk(-x\sin\theta' + z\cos\theta')} \tag{19.21}$$

- At the boundary with the perfect conductor $z = 0$, and we know that E_x must equal zero as well. So, setting $E_x(x,0) = 0$ gives us

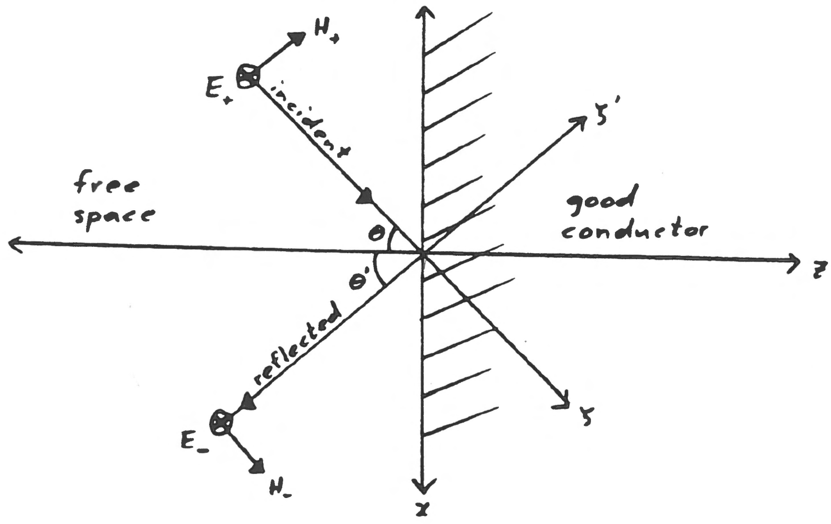
$$E_+ \cos\theta e^{-jkx\sin\theta} = E_- \cos\theta' e^{-jkx\sin\theta'} \tag{19.22}$$

The only way that the complex numbers on both sides of this equality can be equal *for all values of x* (i.e. anywhere along the surface of the conductor) is if they are identical in both magnitude and phase. We conclude, therefore, that

$$k \sin\theta = k \sin\theta' \quad \text{so} \quad \theta = \theta' \quad \text{and} \quad E_+ = E_- \tag{19.23}$$

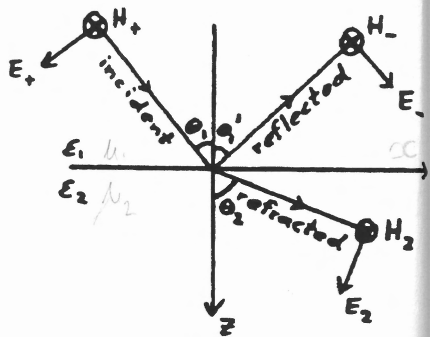
and so we see that the reflection at the perfect conducting surface is *lossless*, and that the angle of reflection to the normal is *equal to the angle of incidence.*

- You should not find it difficult to come to the same conclusion in studying this second diagram, in which the incident and reflected waves both have their **E**-fields directed straight into the paper (perpendicular to the plane of incidence). This is referred to as *transverse electric*, or *perpendicular*, *polarisation.* Since any other orientation of the **E**-field may be regarded simply as a superposition of parallel and perpendicular polarisations, we know that the important conclusions that reflection is *symmetric* and *lossless* will be true, irrespective of what linear polarisation the incident wave has.



19.3 Incidence on a dielectric boundary

- Unlike at the boundary of a perfect conductor, across which a plane wave cannot transmit energy, a travelling wave is not so constrained at the boundary between two dielectric media. At such a boundary, we know from our early consideration of boundary conditions that the tangential components of the **E** and **H** fields are unchanged by the change of medium. Furthermore, there is some transfer of energy from the first medium to the second as the wave crosses the boundary; and yet not all of the incident energy is transmitted because the normal components of **E** and **H** field are not the same in both dielectrics. Hence, we expect to see both *transmission* (known as *refraction*) and also *reflection* in such a situation.



- Following our approach at the perfect conductor boundary, and with reference to the diagram, we can write an equation for the x-components of **E**-field at a point $(x, 0)$ anywhere on the dielectric boundary:

$$E_x(x,0) = -E_+ \cos\theta_1 e^{-jk_1 x \sin\theta_1} + E_- \cos\theta_1' e^{-jk_1 x \sin\theta_1'} \quad (19.24)$$

$$= -E_2 \cos\theta_2 e^{-jk_2 x \sin\theta_2} \quad (19.25)$$

Since the boundary condition dictates that

$$E_{t1} = E_{t2} \quad (19.26)$$

and, since the previous expression must be true irrespective of the value of x (i.e. everywhere on the boundary), we can conclude, similarly to the case of the perfect conductor, that the following terms, known as *phase factors*, must be equal:

$$k_1 \sin\theta_1 = k_1 \sin\theta_1' = k_2 \sin\theta_2 \quad (19.27)$$

$k = \sqrt{\cancel{\omega}\mu\varepsilon} \cdot \omega$

- We can obtain two very useful facts from these equations. The first equality implies, just as with the conducting boundary, that

  $$\theta_1 = \theta_1' \tag{19.28}$$

  so that reflection is symmetric about the normal line, much as we might expect (consider air and water as the two dielectrics, for instance). The second equality, together with several of our earlier definitions, yields

  μ is the same since dielectrics [illegible] from [illegible] non-magnetic.

  $v = \frac{1}{\sqrt{\mu \varepsilon}}$

  $$\frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1} = \frac{k_1}{k_2} = \frac{v_2}{v_1} \approx \sqrt{\frac{\epsilon_1}{\epsilon_2}} \approx \frac{\eta_2}{\eta_1} \tag{19.29}$$

  see constant

  where

  $\frac{v_2}{v_1} = \frac{c/n_1}{c/n_2} = \frac{n_1}{n_2}$

  $$n_x = \frac{c}{v_x} = \sqrt{\frac{\mu_x \epsilon_x}{\mu_o \epsilon_o}} \tag{19.30}$$

  is called the *refractive index* of medium x, and the approximation signs hold because $\mu_1 \approx \mu_2 \approx \mu_o$ for all common media. You should recognise that we have therefore proven *Snell's Law of Refraction* from the standpoint of field theory.

- We can also obtain useful expressions for how much of the incident wave is reflected and how much is refracted at the dielectric boundary. To do this, we define a quantity called the *wave impedance*, which is simply the ratio of the electric and magnetic fields in particular directions. Thus, in the first medium, the wave impedance is

  $$Z_1 = \frac{E_{x+}}{H_{y+}} = -\frac{E_{x-}}{H_{y-}} = \eta_1 \cos\theta_1 \tag{19.31}$$

  Likewise, on the other side of the boundary, we imagine that the wave "sees" a *load* of impedance

  $$Z_L = \frac{E_{x2}}{H_{y2}} \tag{19.32}$$

- Now, the continuity of the tangential components of **E** and **H** at the boundary gives us

$$E_{x+} + E_{x-} = E_{x2} \quad \text{and} \quad H_{y+} + H_{y-} = H_{y2} \tag{19.33}$$

It should be clear that the second of these two equations gives us the equation below. (We will then substitute the first equation into this new equation in two different ways to quantify the reflection and refraction at the boundary).

$$\frac{E_{x+}}{Z_1} - \frac{E_{x-}}{Z_1} = \frac{E_{x2}}{Z_L} \tag{19.34}$$

1. Using the first equation to substitute for E_{x2} gives us

$$\frac{E_{x+}}{Z_1} - \frac{E_{x-}}{Z_1} = \frac{E_{x+}}{Z_L} + \frac{E_{x-}}{Z_L} \tag{19.35}$$

so

$$E_{x+}\left[\frac{1}{Z_1} - \frac{1}{Z_L}\right] = E_{x-}\left[\frac{1}{Z_1} + \frac{1}{Z_L}\right] \tag{19.36}$$

and from this we can define the *reflection coefficient*, ρ, which turns out to be

$$\rho \equiv \frac{E_{x-}}{E_{x+}} = \frac{\frac{1}{Z_1} - \frac{1}{Z_L}}{\frac{1}{Z_1} + \frac{1}{Z_L}} = \frac{Z_L - Z_1}{Z_L + Z_1} \tag{19.37}$$

Hence, if we know the wave impedances of the two media, we can predict what fraction of the incident wave is reflected at the boundary.

2. In a similar way, using the first equation to substitute for E_{x-} will give

$E_{x-} = E_{x2} - E_{x+}$

$$\frac{E_{x+}}{Z_1} - \frac{(E_{x2} - E_{x+})}{Z_1} = \frac{E_{x2}}{Z_L} \tag{19.38}$$

which quickly gives

$$\frac{2E_{x+}}{Z_1} = E_{x2}\left[\frac{1}{Z_L} + \frac{1}{Z_1}\right] \tag{19.39}$$

so we can then define the *transmission coefficient*, τ, as

$$\tau \equiv \frac{E_{x2}}{E_{x+}} = \frac{\frac{2}{Z_1}}{\frac{1}{Z_L} + \frac{1}{Z_1}} = \frac{2Z_L}{Z_1 + Z_L} \tag{19.40}$$

This can be used to predict how much of the incident wave is refracted at the boundary.

- You will certainly have noticed the similarity between these equations and those which you have met in your study of transmission lines. This should confirm for you that the *guided* transmission of electromagnetic waves that one studies in transmission line theory is described by field theory just as effectively as the case in which the waves propagate, unguided, through the surrounding medium.

Part 4

Sample Questions

Test 1 Specimen Questions

1995 Test 1

Attempt **any three** questions, and state all your final answers in **engineering units**. Credit is given for showing all of your working carefully. The time allowed is **50 minutes**.

60 marks will gain 100%

1995 Test1 Question1 (20 MARKS)

(a) Show that the electric field intensity due to a ring of positive charge, of charge density ρ_l and radius a, perpendicular to the z-axis, is given at any point on the z-axis with co-ordinate z by

$$\mathbf{E} = \frac{\rho_l \, a z}{2\epsilon(a^2 + z^2)^{\frac{3}{2}}} \hat{\mathbf{e_z}} \qquad [8]$$

(b) Hence state the magnitude and the direction of the electric flux density vector, $\mathbf{D}$, at a distance of z = 10cm up the axis from the centre of a charge ring of radius 10cm, assuming a charge density of $\rho_l = 2\mathrm{C\,m^{-1}}$. Be careful to state your answer with units. [6]

(c) Make a neat sketch of the electric field in the vicinity of the charge ring, showing the behaviour of the field lines both on and off the ring's axis. What is the appearance of the field at a distance from the ring that is much greater than its radius? [6]

1995 Test1 Question2 (20 MARKS)

(a) Calculate the electric field intensity between the two cylinders of a coaxial line separated by a medium of permittivity ϵ. Use Gauss's Law, with a special gaussian cylindrical surface of length l and radius r to do this. You should take the charge per unit length on the inner cylinder as ρ_l, and should obtain the answer

$$E_r = \frac{\rho_l}{2\pi\epsilon r} \qquad [5]$$

(b) Now calculate $\nabla \cdot \mathbf{D}$ in the region between the cylinders. Is the result what you might have expected? Explain. [5]

(c) The inner cylinder is at a potential of 1V and has a radius of 1.831cm. The outer cylinder is at a potential of 5V and has a radius of 4.978cm. Use this information to solve Laplace's equation between the two cylinders, to arrive at the following formula for the distribution of the potential at any radius r:

$$\Phi = 4\ln|r| + 17 \qquad [5]$$

(d) From the answer to part (c), find the radii of the circles of equipotential at 2V, 3V and 4V. These circles have been drawn to scale for you on the accompanying full-scale diagram. Add equally-spaced field lines to this diagram, and hence deduce the capacitance per metre of the coaxial line.

[5]

1995 Test1 Question3 (20 MARKS)

(a) For two point charges q and $-q$ at $(d/2, 0, 0)$ and $(-d/2, 0, 0)$ respectively, find the potential for any point (x, y, z) and from this derive the electric field. Check the results by adding vectorially the electric field from the individual charges. [4,4,2]

(b) Find the electric field and charge density as functions of x for a space-charge-limited, parallel-plane diode with potential variation given by

$$\Phi = V_o \left(\frac{x}{d}\right)^{\frac{4}{3}} \qquad [5,5]$$

1995 Test1 Question4 (15 MARKS)

(a) Show that a parallel-plate capacitor with plates of area A and separation d, charged to $\pm Q$ and at a charge density of ρ_S Cm^{-2}, has a flux density between the plates of magnitude $D = \rho_S$. State clearly any assumptions that are made in your explanation. [4]

(b) Use the definitions of potential and capacitance to show:

- (i) that the voltage between the plates is $V_c = \rho_S\, d/\epsilon$; [5]
- (ii) that the capacitance of the parallel plates is $C = \epsilon A/d$ [5]

(c) Then, given the result that the energy stored in an infinitesimal volume dV of an electric field $\mathbf{E}$ of flux density $\mathbf{D}$ is

$$dU_c = \frac{1}{2}\mathbf{D} \cdot \mathbf{E}\, dV,$$

show that the energy stored in the whole field of the capacitor is

$$U_c = \frac{1}{2} C V_c^2$$

Again, state clearly any assumptions made in your analysis. [6]

1996 Test 1

Attempt **any two or three** questions, and state all your final answers in **engineering units**. Credit is given for showing all of your working carefully. The time allowed is **50 minutes**.

50 marks will gain 100%

1996 Test1 Question1 (20 MARKS)

(a) Show by integration of the contributions from each differential charge that the electric field intensity due to an infinite straight line of positive charges, of uniform charge density ρ_l distributed along the z-axis, is given in cylindrical co-ordinates at a distance r from the line by:

$$\mathbf{E} = \frac{\rho_l}{2\pi\epsilon r}\hat{\mathbf{r}}$$

where ϵ is the permittivity of the surrounding medium and $\hat{\mathbf{r}}$ is the unit radial vector. **[7]**

(b) Verify the result given above by applying Gauss's Law to a carefully chosen gaussian surface near the line of charges. Present your argument in terms of a clearly-labelled, neat diagram, and pay particular attention to correct notation. Explain your steps in words where necessary. **[7]**

(c) Although the result in part (a) above refers to a line of *static* charge, it is also adequate for finding the electric field due to a steady current in a wire.

(i) Explain briefly why this is so. **[3]**

(ii) Make a sketch of the electric field near a wire carrying a steady current. **[3]**

1996 Test1 Question2 (20 MARKS)

A parallel-plate capacitor consists of a pair of large conducting plates which are separated by a dielectric medium of relative permittivity $\epsilon_r = 3.0$. The plates are perpendicular to the z-axis, as shown in the upper drawing on page 6, and each plate is a square of side 5cm (i.e. of area 0.0025m^2). The plate separation is $d = 5\text{mm}$.

At a given instant in time, currents in the wires leading to the capacitor have produced an accumulation of positive charge on the upper plate and negative charge on the lower plate, so that the plates have charge density ρ_s Cm^{-2} and $-\rho_s$ Cm^{-2} respectively.

(a) Given that the electric field intensity due to each plate has magnitude $E = \frac{\rho_s}{2\epsilon}$, briefly state (with a reason) the total electric field intensity vector in the region between the two plates. **[2]**

(b) From your answer to part (a) above, show that the potential function between the capacitor plates, as a function of the distance from the upper plate, is $\Phi = 6 - \frac{\rho_s}{\epsilon} z$, using the fact that at the upper plate (where $z = 0$) the potential is 6V. **[4]**

(c) Given also that the lower plate is at ground potential (0V), use the result given in part (b) above to state the total charge Q on the upper plate. **[4]**

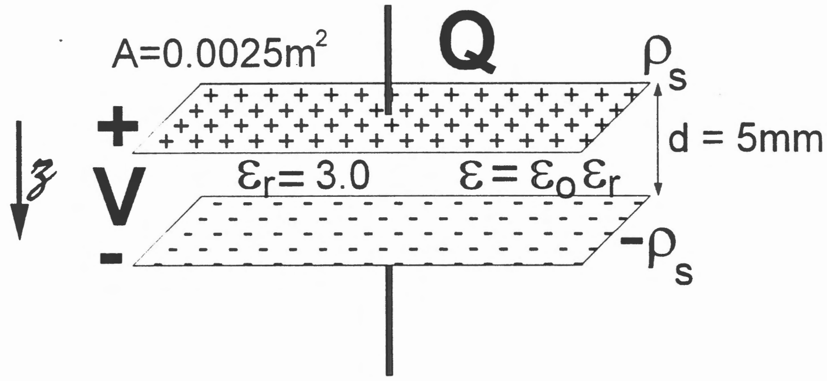
(d) A cross-sectional view of the parallel-plate capacitor is drawn to scale on page 6, (which you must hand in, if you answer this question):

(i) Add to this drawing the equipotential lines for $\Phi = 2\text{V}$ and $\Phi = 4\text{V}$, and also add appropriate field lines to complete a graphical field map. **[3]**

(ii) Calculate the total capacitance of the plates from your field map, ignoring the effects of fringing (i.e. consider *only* the field lines and the equipotentials that are directly between the plates). **[3]**

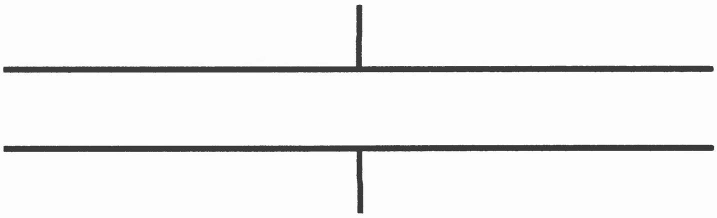
(iii) Compare your answer to part (ii) above with the capacitances as calculated from the well-known expressions:

$$C = \frac{A}{d}\epsilon \quad \text{and} \quad C = \frac{Q}{V}$$ **[2,2]**



Question 2(d) **NAME:**__________________

(i)



(ii)

(iii)

1996 Test1 Question3 (20 MARKS)

A steady current in a long wire gives rise to an electric field $\mathbf{E} = \frac{\rho_l}{2\pi\epsilon r}\hat{\mathbf{r}}$ where ρ_l is the net charge density of the current carriers which are uniformly distributed along the wire, ϵ is the permittivity of the surrounding medium, and r is the radial distance, in cylindrical co-ordinates, from the central z-axis of the wire.

(a) Show that the divergence of the electric flux density vector is equal to the net charge density both inside and outside the wire. **[3,3]**

(b) The wire can be viewed as an ohmic resistor of conductivity σ. State:

(i) Ohm's Law, in terms of field quantities;

(ii) The Principle of Continuity in differential form;

and use these equations to show that the potential function Φ in the vicinity of the wire will obey Laplace's Equation:

$$\nabla^2\Phi = 0$$ **[2,2,2]**

(c) In the solution of Laplace's Equation in this case, explain very briefly why it is acceptable to neglect:

(i) the tangential (ϕ) component of the potential;

(ii) the longitudinal (z) component of the potential. **[1,1]**

(d) Solve Laplace's Equation for this case, with the boundary condition that at a distance r_o from the wire the potential is zero, and hence show that the potential function around the wire as a function of the radial distance r is

$$\Phi = A\ln|r| - A\ln|r_o|$$

where A is a constant. **[4]**

(e) Calculate the potential function around the wire directly from the given expression for the electric field intensity, and hence show that the constant A in part (d) above is given by

$$A = -\frac{\rho_l}{2\pi\epsilon}$$ [2]

1996 Test1 Question4 (20 MARKS)

(a) Write down the relationship between electric field intensity and electrostatic potential both in its integral form and in its differential form. Start from this relationship to explain in careful detail why the equipotentials are everywhere perpendicular to the field lines in an electrostatic field. **[1,1,3]**

(b) Write down Gauss's Law both in its integral and in its differential form, and show that these two expressions of Gauss's Law are equivalent. **[1,1,3]**

(c) Give a simple verbal statement of what is meant by the divergence of a vector quantity, and then derive from first principles the expression for $\nabla \cdot \mathbf{D}$ in the cartesian co-ordinate system, namely:

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

[8]

(d) A research engineer calculates the electrostatic potential function at a moment in time between the stator and the rotor of a motor. The function is very complicated, and the researcher is obliged to find it by using an expensive computer package which applies elegant numerical methods to the boundary conditions. These boundary conditions have to be input for the particular case of the motor in question. Once the computer completes the calculation, what does the engineer know for certain about the solution for the potential function that the computer gives?

[2]

Test 2 Specimen Questions

1995 Test 2

Attempt **any two** questions in full plus **part (a) only** of the other two questions. State all final answers in **engineering units**. Credit is given for showing all of your working carefully. The time allowed is **50 minutes**. 48 marks will gain 100%

1995 Test2 Question1 (20 MARKS)

(a) State the Biot-Savart Law, and provide a diagram which illustrates all of the seven variables that are connected by this Law. [4]

(b) Use the Biot-Savart Law to calculate the magnetic field vector (magnitude and direction) at a point distance r from a straight wire of infinite length which carries a dc current I_o. You may take as given the integral

$$\int_a^a \frac{dz}{(r^2+z^2)^{\frac{3}{2}}} = \frac{2}{r^2\sqrt{(r/a)^2+1}} \qquad [5]$$

(c) State Ampère's Circuital Law, and explain carefully (5-10 lines plus a diagram) how this Law yields the same result as your answer to part (b) for the magnetic field of an infinite wire carrying dc current I_o. [5]

(d) From the defining equation for magnetic flux density, $d\mathbf{f} = I d\mathbf{l} \times \mathbf{B}$, use the result of parts (b) and (c) to calculate the magnitude and direction of the force per metre length between two parallel wires 1m apart, each carrying a current of 1A in the same direction. *(This is often regarded as the primary definition of the ampère in the SI system).* [6]

1995 Test2 Question2 (20 MARKS)

(a) Make a sketch of the magnetic fields lines in the vicinity of a parallel-microstrip transmission line. [4]

(b) A microstrip transmission line consists of two long parallel plates of width w, copper thickness h and separation d. The plates are separated by a dielectric of magnetic permeability μ. Listing **three** important and distinct assumptions that are made in your analysis, use Ampère's Circuital Law to show that the inductance per unit length of such a transmission line is

$$L = \frac{\mu d}{w} \quad [\mathrm{Hm}^{-1}] \qquad [7]$$

(c) With reference to your sketch in part (a), explain what you would expect to happen to an imaginary "curl meter paddle" placed exactly between the two conducting plates of the transmission line discussed in part (b). Can you justify this expectation in terms of a mathematical expression for the curl of magnetostatic field? [5]

(d) Show that, if the magnetic field between the parallel plates is written H_o and current I flows in them, then, given the assumptions listed in part (b),

$$\int_V \frac{\mu}{2} H_o^2 \, dV = \frac{1}{2} L I^2 \qquad [4]$$

1995 Test2 Question3 (20 MARKS)

(a) What is meant by describing a magnetic material as

(i) linear (ii) isotropic

It is acceptable to give each answer either as a simple equation or as a very short explanation. In either case, give your answer in terms of the quantities **B**, **H** and μ. [4]

(b) Starting with the expression for stored magnetic energy,

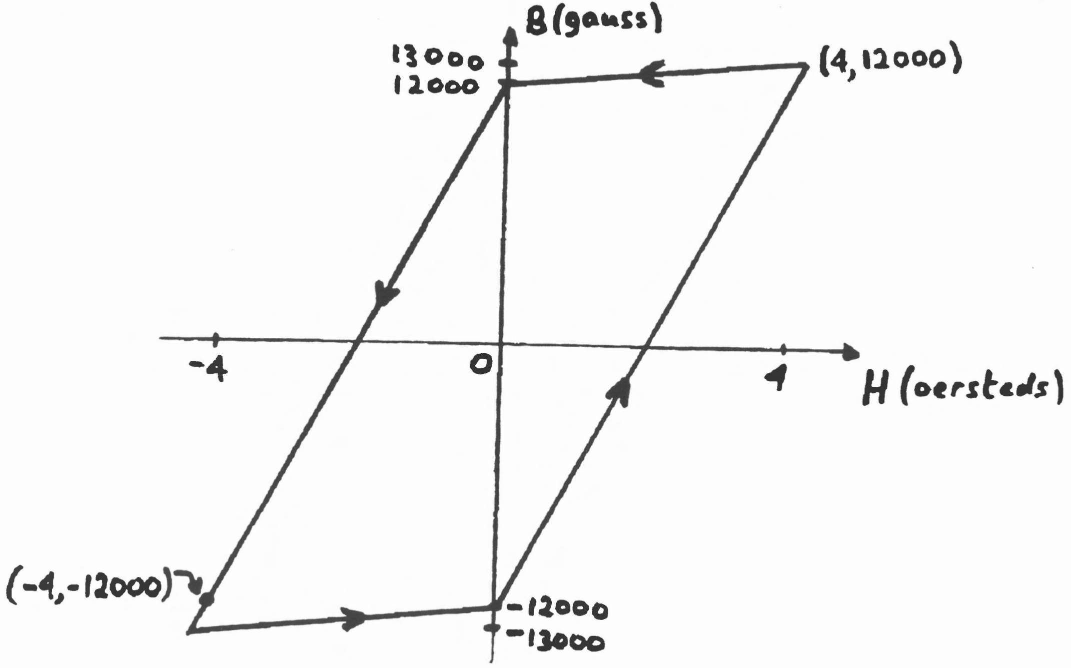
$$dU_H = \int_V \mathbf{H} \cdot d\mathbf{B}\, dV$$

explain how and why it is possible to estimate the energy dissipated in a unit volume of a *non-linear*, *isotropic*, *hysteretic* material, given a plot of the BH response of the material over a complete cycle of periodic magnetic field variation. [8]

(c) The diagram below is an approximation of the hysteretic loop for fairly pure iron. Use the result that you gave in part (b) to show that, in a sample of this material of volume $2\pi \times 10^{-3}\text{m}^3$, 5J of energy is dissipated per cycle of the periodic magnetic field variation. You are given that

- 1 gauss= 0.1mT
- 1 oersted= $250/\pi\,\text{Am}^{-1}$
- 1 tesla= $1\text{Nm}^{-1}\text{A}^{-1}$

[8]



1995 Test2 Question4 (20 MARKS)

(a) Write down Maxwell's four equations for time-varying electromagnetic fields, in their differential form. [4]

(b) EITHER: Explain in detail how Faraday's Law of Electromagnetic Induction is used to obtain the expression for the curl of electric field in the time-varying case, as a generalisation of the static case in which $\nabla \times \mathbf{E} = 0$;

OR: Explain in detail how the concepts of *continuity* and of *displacement current* are used to obtain the expression for the curl of magnetic field in the time-varying case, as a generalisation of the static case in which $\nabla \times \mathbf{H} = \mathbf{J}$. [8]

(c) Show that your four answers to part (a) are equivalent to the integral forms:

1. $\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV$
2. $\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$
3. $\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$
4. $\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{\partial}{\partial t} \int_S \mathbf{D} \cdot d\mathbf{S}$

and state very briefly what each equation expresses, in either its differential or its integral form. [8]

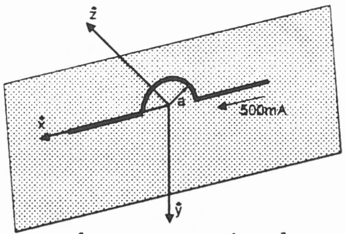
1996 Test 2

Attempt **any three** questions. Credit is given for showing all of your working carefully. The time allowed is **50 minutes.**

45 marks will gain 100%

1996 Test2 Question1 (15 MARKS)

On part of a certain printed circuit board, a dc current of 500mA flows in a semicircular loop of radius a = 4cm, as illustrated below. Show that the magnetic field intensity due to the current in the semicircle, measured at a point with distance z from the radial centre of the semicircle along the z-axis of the semicircle (this axis being perpendicular to the printed circuit board), has z-component



$$H_z = \frac{Ia^2}{4(a^2+z^2)^{\frac{3}{2}}}$$

[4]

and y-component

$$H_y = \frac{Iaz}{2\pi(a^2+z^2)^{\frac{3}{2}}}$$

[4]

and x-component

$$H_x = 0$$

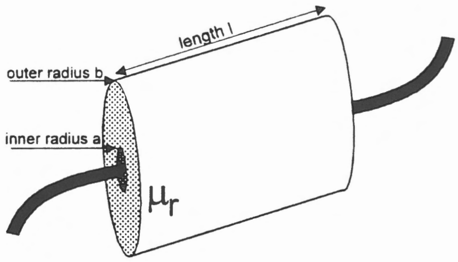
[1]

Hence state the magnitude and the angle to the z-axis of the magnetic field intensity vector at a point with distance z = 3cm along the z-axis from the semicircle's radial centre.

[3,3]

1996 Test2 Question2 (15 MARKS)

It is common practice in some industrial applications to suppress unwanted signal spikes in a conductor by threading the conductor through a cylindrical bead as shown below.



(a) Explain carefully how you would calculate the inductance due to the bead in this configuration, and derive a formula for this inductance in terms of the bead's dimensions, a, b and l, and in terms of the permeability, μ, of the material of which the bead is made.

[5]

(b) Use the formula which you derived in part (a) to calculate the inductance due to an actual bead which has an inner radius of 1mm, an outer radius of 2.15mm, a length of 7.2mm, and which is fabricated from a material of relative permeability $\mu_r = 900$.

[4]

(c) The actual inductance of this bead has been measured accurately, and was found to be 1μH. Can you explain any small discrepancy between this measured value and your calculated answer above?

[3]

(d) Describe how and why the bead acts as required in the suppression of unwanted signal spikes.

[3]

1996 Test2 Question3 (15 MARKS)

(a) Show clearly that electrostatic fields have zero curl.

[3]

(b) Starting from the basic definition of curl, use Ampère's circuital law to derive an expression for the curl of magnetic field intensity.

[5]

(c) Magnetic flux density may also be found from the curl of magnetic vector potential. Use this fact together with Stokes's theorem and the divergence theorem to show clearly that magnetostatic fields have zero divergence.

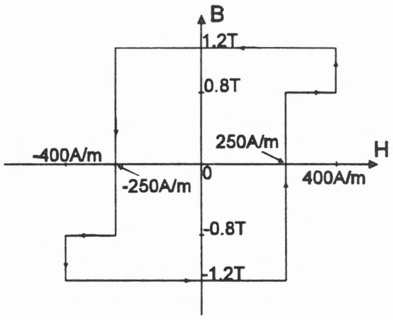
[5]

(d) Carefully explain the consequences of the relationship which you derived in part (c).

[2]

1996 Test2 Question4 (15 MARKS)

The diagram below approximates the hysteretic loop for a 100cm^3 sample of magnetic material that is the core of a multitap transformer operating at 50Hz and converting 3A (rms) at 220V (rms) to various output voltages.



(a) Identify

- (i) the coercive force
- (ii) the remanence

of the transformer core.

[1,1]

(b) Starting with the expression for stored magnetic energy,

$$dU_H = \int_V \mathbf{H} \cdot d\mathbf{B}\, dV$$

explain how and why it is possible to estimate the energy dissipated in a unit volume of a *non-linear*, *isotropic*, *hysteretic* material, given a plot of the BH response of the material over a complete cycle of periodic magnetic field variation.

[5]

(c) Estimate the power loss in watts due to hysteresis in the transformer core, and hence state the efficiency of the transformer, assuming that 2% of the input power is dissipated in the copper windings and that the only other loss is due to hysteresis.

[5,3]

Specimen Exam Questions

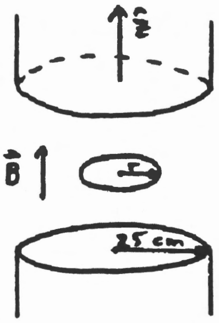
Q1 (20 MARKS)

(a) State Faraday's Law of Electromagnetic Induction, and give a reasoned explanation, in terms of magnetic flux linkage, of what the Law means.

[8]

(b) A large electromagnet has circular poles, each of radius 25 cm as shown in this diagram. You may assume that the magnetic field is uniform between these poles in the $\hat{\mathbf{z}}$-direction as shown, and zero elsewhere. Use Faraday's Law in the vicinity of a loop of radius r to show that the electric field intensity vector between the poles has circumferential component only, and that this component is given by E_ϕ, where

$$2\pi r E_\phi = -\pi r^2 \frac{\partial B_z}{\partial t}$$



[4]

(c) Explain why the **E**-field has greatest magnitude where $r = 25$ cm.

[3]

(d) It is proposed to raise the magnitude of the **B**-field from zero to some maximum value, B_{max}. This is to take place over a period of 250 ns, in such a way that the air between the poles will not break down. At sea level air breaks down at about 3000 kV m^{-1}. Find the maximum value to which the **B**-field can be raised under these circumstances.

[5]

Q2 (17 MARKS)

(a) A coaxial cable consists of a solid inner cylinder of radius a and a thin outer cylindrical conductor of radius b. The inner cylinder carries a net charge of density $\rho\,\mathrm{Cm}^{-3}$. The length of the cable is very much greater than the radius of either cylinder. Use Gauss's Law, with an appropriate gaussian surface and a suitable co-ordinate system, to find the electric flux density vector both inside and outside the inner cylinder. Then calculate the divergence of electric flux density both inside and outside the inner cylinder (but inside the outer cylinder), and comment on your results.

[5]

(b) Current flows in the outer conductor in one direction and back in the other direction in the inner conductor. Use Ampère's Circuital Law to explain clearly an advantage of carrying current in a coaxial structure rather than, for example, simply using two separate wires.

[2]

(c) The inner cylinder is at a potential of 2V with respect to the earthed outer cylinder. The radii of the two cylinders are $a = 2.479\,\mathrm{mm}$ for the inner, and $b = 6.738\,\mathrm{mm}$ for the outer. Use this information to solve Laplace's equation in the region between the two cylinders, in order to arrive at the following formula for the distribution of the potential at any radius r:

$$\Phi = -2\ln|r| - 10$$

[5]

(d) From the answer to part (c), find the radii of the cylinders of equipotential at 0.5V, 1.0V and 1.5V. Draw a scale diagram (10 times actual size) of the cross section of the conductors and of the equipotentials. Use this diagram to find the capacitance per metre of the coaxial cable, given that the separating medium has a relative permittivity of $\epsilon_r = 2$.

[5]

Q3 (20 MARKS)

(a) Explain briefly what is meant by:

1. Conduction Current
2. Convection Current
3. Displacement Current

In each case you should also supply an equation which defines the stated quantity.

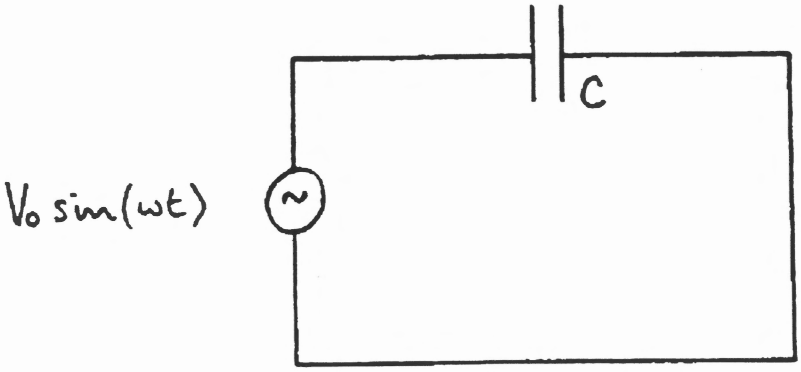
[6]

(b) Show how the concept of displacement current is used in the derivation of a full statement of Ampère's Circuital Law:

$$\oint \mathbf{H} \cdot d\mathrm{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{\partial}{\partial t} \int_S \mathbf{D} \cdot d\mathbf{S}$$

[7]

(c) Use the law you derived in part (b) to show that the displacement current density within the capacitor in the circuit below is equal to the conduction current density along the connecting wires of the circuit.



[7]

Q4 (17 MARKS)

(a) Given that the magnetic field due to a current I flowing in a circular loop of radius a that lies in the xy-plane is purely z-directed and that, at any point z along the axis of the loop, it has value

$$H_z = \frac{Ia^2}{2(a^2 + z^2)^{\frac{3}{2}}}$$

show that the magnetic field of a solenoid of n turns per metre, carrying current I, is

$$H_z = \frac{nI}{\sqrt{(\frac{a}{b})^2 + 1}}$$

where a is the radius of the solenoid and b is half its length.

[4]

(b) Using the result from part (a), discuss the magnitude and direction of the magnetic field of a solenoid of infinite length and of radius a:

(i) on the axis of the solenoid;

(ii) within the solenoid but off the axis;

(iii) outside the solenoid.

In each case, provide a coherent proof for your stated conclusions and explain any assumptions which are made in your analysis.

[2,2,2]

turn over for part (c)

(c) A coiled current-carrier has radius $a = 5\,\mathrm{mm}$ and length $b = 1\,\mathrm{cm}$. It is composed of $N = 1000$ turns of fine copper wire that are wound about a former of relative permeability $\mu_r = 125$. The wire carries a current of $I = 500\,\mathrm{mA}$.

  (i) Derive a general expression in terms of a, b, N, μ_o, μ_r and I for the energy stored per one-metre length of such a coil, and hence find the energy stored in the magnetic field of the particular coil described;

  [4]

  (ii) Show that the inductance of the particular coil is very nearly 1 H.

  [3]

Q5 (20 MARKS)

(a) Use Maxwell's equations to show that, for an electromagnetic wave travelling in free space in the z-direction,

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

and

$$-\frac{\partial H_y}{\partial z} = \epsilon \frac{\partial E_x}{\partial t}$$

[5]

(b) Show that, if **E** and **H** both vary sinusoidally, the equations in part (a) give rise to the one-dimensional Helmholtz equation:

$$\frac{d^2 \tilde{E}_x}{dz^2} = -\omega^2 \mu \epsilon \tilde{E}_x$$

Show that a solution of this equation is

$$\tilde{E}_x = c_1 e^{-jkz} + c_2 e^{jkz}$$

where c_1 and c_2 are constants and $k = \omega\sqrt{\mu\epsilon}$

[5]

(c) Hence show that $\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\epsilon}}$ for the positively-travelling wave solution to Maxwell's equations in free space.

[5]

(d) Make a sketch of an electromagnetic wave travelling in the z-direction in free space. Show clearly the directions of **E** and **H** at all points along the wave, and indicate the ratio of the magnitudes of **E** and **H** as well.

[5]

Q6 (17 MARKS)

(a) Explain why a high-frequency signal being beamed to a satellite (which would typically have a wavelength of a few centimetres) is well-described over most of its path as a plane wave in free space.

[2]

(b) Write down Maxwell's equations for free space, and use them to derive the one-dimensional wave equation that governs the x-component of electric field, E_x, in a plane wave propagating through free space in the z-direction. Explain each step in your derivation with care.

[7]

(c) Show that

$$E_x = \cos(t - \sqrt{\mu_o \epsilon_o}\, z) + \cos(t + \sqrt{\mu_o \epsilon_o}\, z)$$

is a solution to the equation that you derived in part (b).

[2]

(d) With reference to the solution suggested in part (c):

(i) show that a signal beamed to a satellite propagates at a velocity close to that of light in a vacuum;

[3]

(ii) describe the effects that may be expected when the signal leaves the atmosphere of the earth.

[3]

Q7 (20 MARKS)

(a) Starting from the standpoint of field theory, show that the voltage drop across a purely inductive element through which a current $i(t)$ flows is given at any instant by

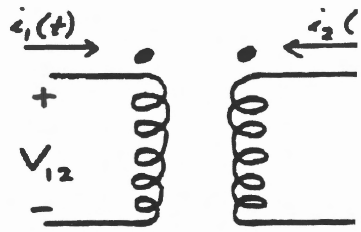
$$v = L\frac{di(t)}{dt}$$

as assumed in circuit theory. Include in your explanation a working definition for the quantity L.

[5]

(b) Two coils, through which currents $i_1(t)$ and $i_2(t)$ flow respectively, are placed in close proximity to each other as shown in this diagram. State clearly what is meant by *magnetic flux linkage*, ψ_m, and *mutual inductance*, M, and show that the voltage induced across the first coil by the current in the second is

$$V_{12} = M\frac{di_2(t)}{dt}$$



[5]

(c) Derive an expression for the inductance per unit length of a coaxial cable. Show all your working clearly. You may assume that the coaxial cable consists of two excellent conductors of radii a and b, and that it has negligible internal inductance. The separating medium has permeability μ.

[5]

(d) Using your answer to part (c), show that the external inductance per metre of a coaxial transmission line in which the inner and outer conductors are separated by a medium of relative permeability μ_r =1.005 is 2 nH if the outer conductor has inner radius 1.01 times the radius of the inner conductor.

[5]

Q8 (20 MARKS)

(a) Either by dimensional analysis, or else arguing from the standpoint of field theory, show that the expression

$$\mathbf{J} = \sigma \mathbf{E}$$

is a statement of Ohm's Law that leads directly to the familiar form in circuit analysis, namely

$$V = IR$$

if a lumped element of length l, conductivity σ, cross-sectional area A and resistance

$$R = \frac{l}{\sigma A}$$

is considered.

[4]

(b) Explain what is meant by *continuity*, and why it is adequately described in a good conductor by

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = 0$$

[3]

(c) Use Ohm's Law, Maxwell's equations and the principle of continuity to show that a sinusoidally-varying current density in a material of permeability μ and conductivity σ is governed by the equation

$$\nabla^2 \tilde{\mathbf{J}} = j\omega\mu\sigma\tilde{\mathbf{J}}$$

[6]

(d) Solve the equation in part (c), and hence explain what is meant by the concept of *skin depth* in a good conductor.

[7]